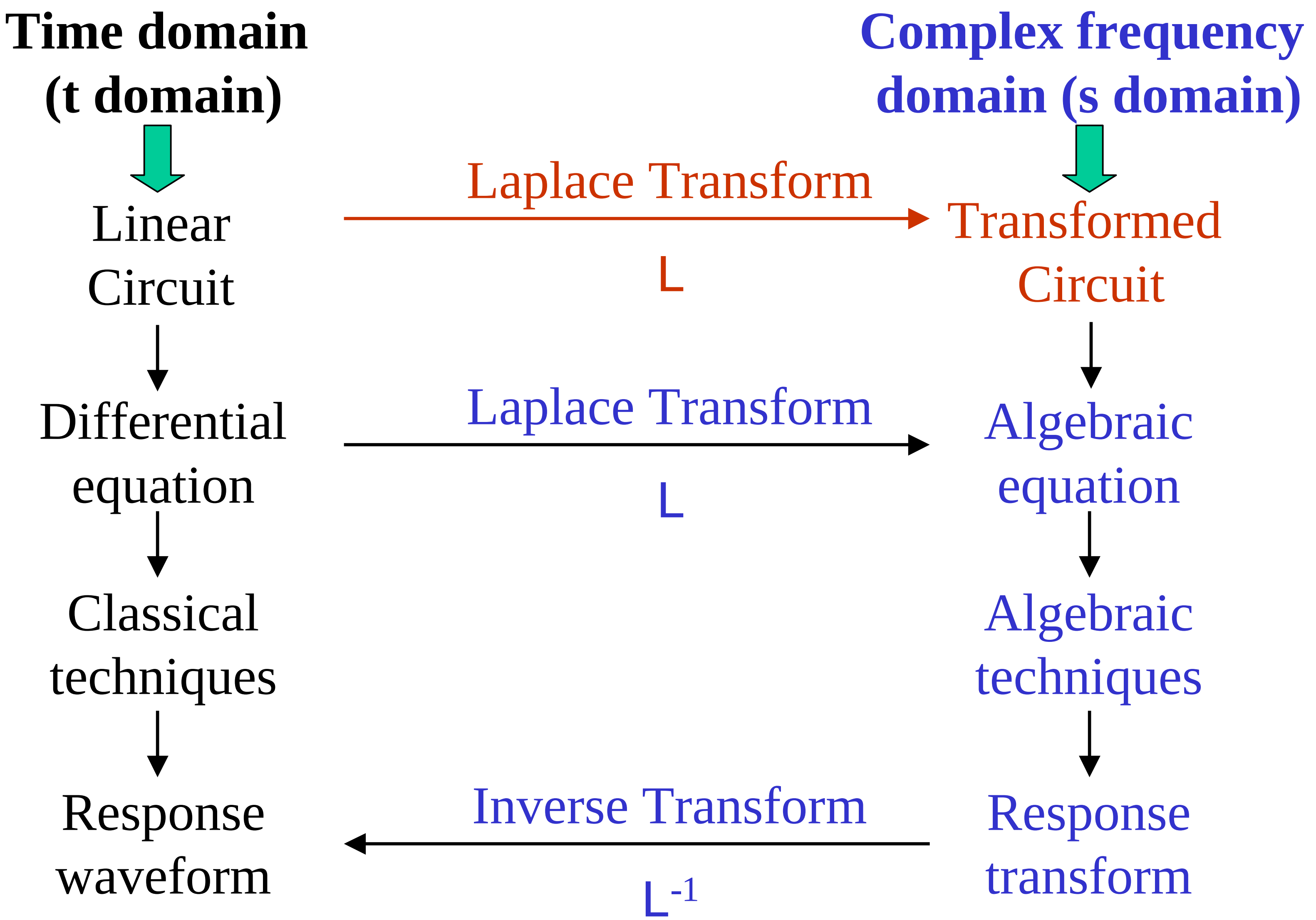
S-Domain Analysis

s-Domain Circuit Analysis

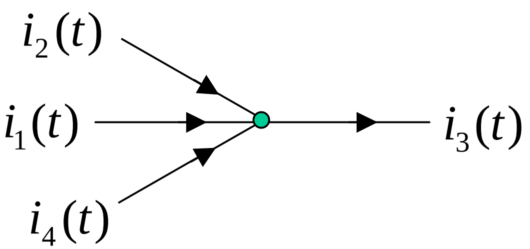


Kirchhoff's Laws in s-Domain

t domain

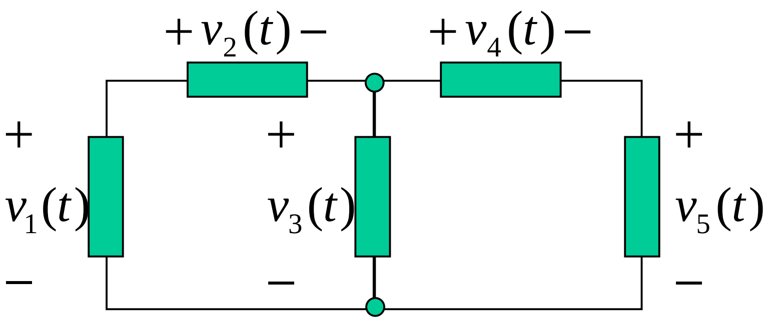
s domain

Kirchhoff's current law (KCL)



$$i_1(t)+i_2(t)-i_3(t)+i_4(t)=0 \qquad I_1(s)+I_2(s)-I_3(s)+I_4(s)=0$$

Kirchhoff's voltage law (KVL)



$$-v_1(t)+v_2(t)+v_3(t)=0 \qquad -V_1(s)+V_2(s)+V_3(s)=0$$

Signal Sources in s Domain

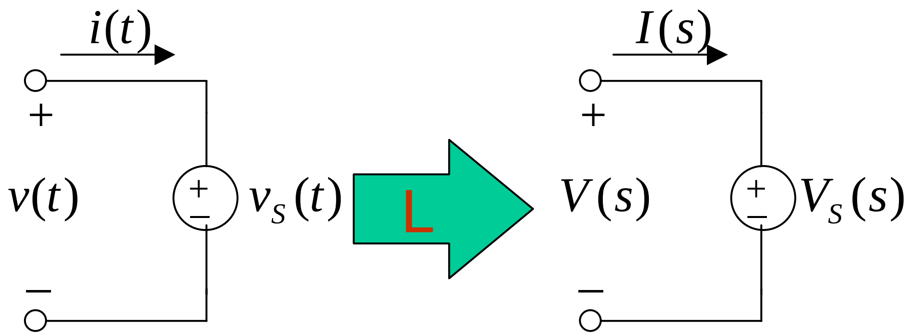
t domain

s domain

Voltage Source:

$v(t) = v_S(t)$

$i(t) = \text{depends on circuit}$



Voltage Source:

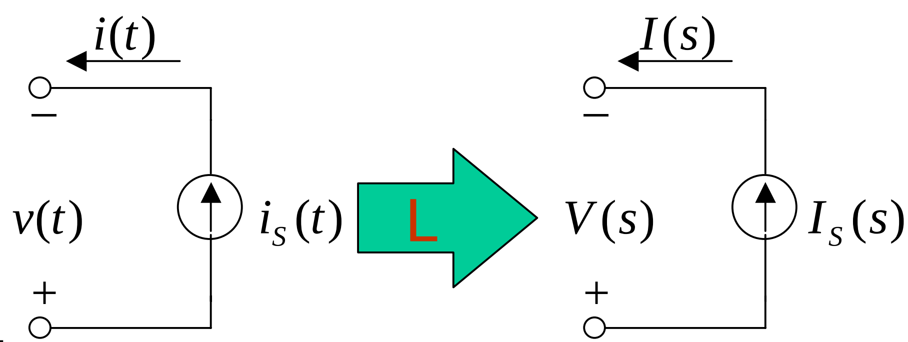
$V(s) = V_S(s)$

$I(s) = \text{depends on circuit}$

Current Source:

$i(t) = i_S(t)$

$v(t) = \text{depends on circuit}$



Current Source:

$I(s) = V_S(s)$

$V(s) = \text{depends on circuit}$

Time and s-Domain Element Models

Impedance and Voltage Source for Initial Conditions

Time Domain

Resistor:

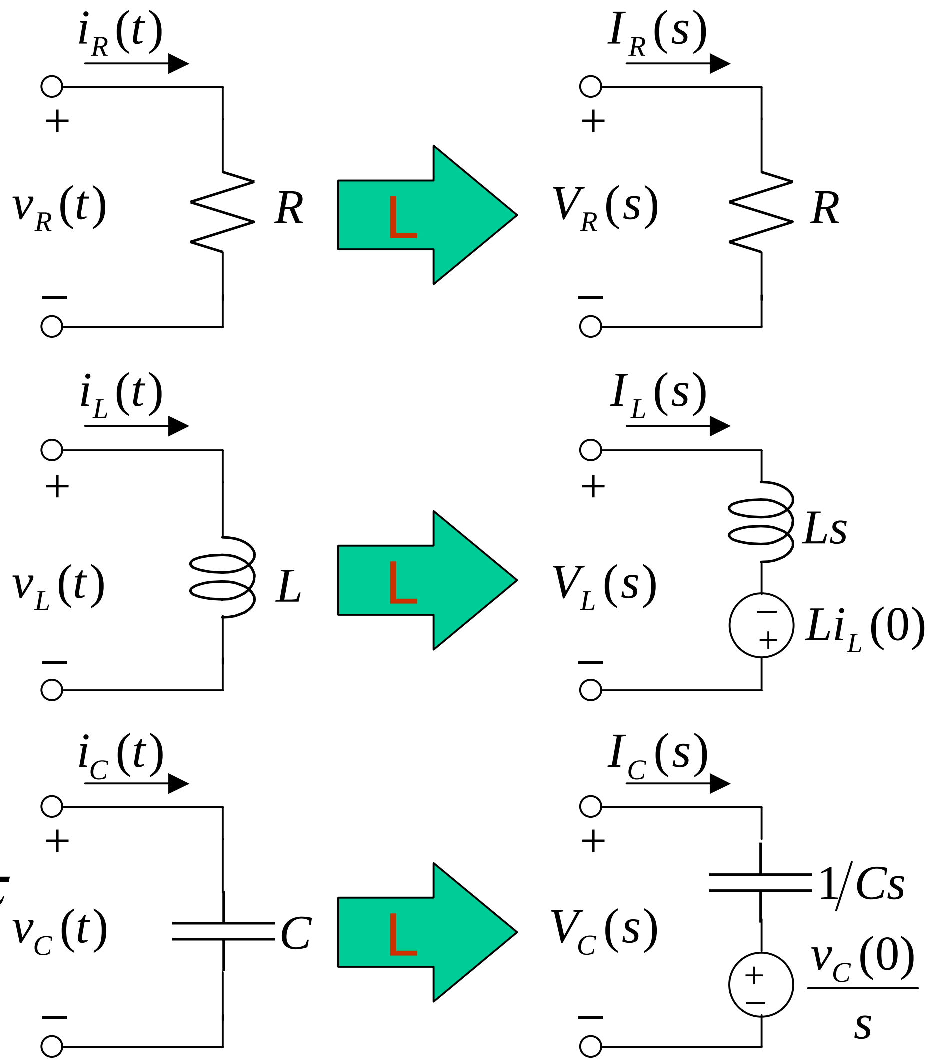
$$v_R(t) = Ri_R(t)$$

Inductor:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

Capacitor:

$$v_C(t) = \frac{1}{C}\int_0^t i_C(\tau)d\tau + v_C(0)$$



s-Domain

Resistor:

$$V_R(s) = RI_R(s)$$

Inductor:

$$V_L(s) = LsI_L(s) - Li_L(0)$$

Capacitor:

$$V_C(s) = \frac{1}{Cs}I_C(s) + \frac{v_C(0)}{s}$$

Impedance and Voltage Source for Initial Conditions

- Impedance Z(s)

$$Z(s) = \frac{\text{voltage transform}}{\text{current transform}}$$

with all initial conditions set to zero

- Impedance of the three passive elements

$$Z_R(s) = \frac{V_R(s)}{I_R(s)} = R$$

$$Z_L(s) = \frac{V_L(s)}{I_L(s)} = Ls \qquad \text{with } i_L(0) = 0$$

$$Z_C(s) = \frac{V_C(s)}{I_C(s)} = \frac{1}{Cs} \qquad \text{with } v_C(0) = 0$$

Time and s-Domain Element Models

Admittance and Current Source for Initial Conditions

Time Domain

Resistor:

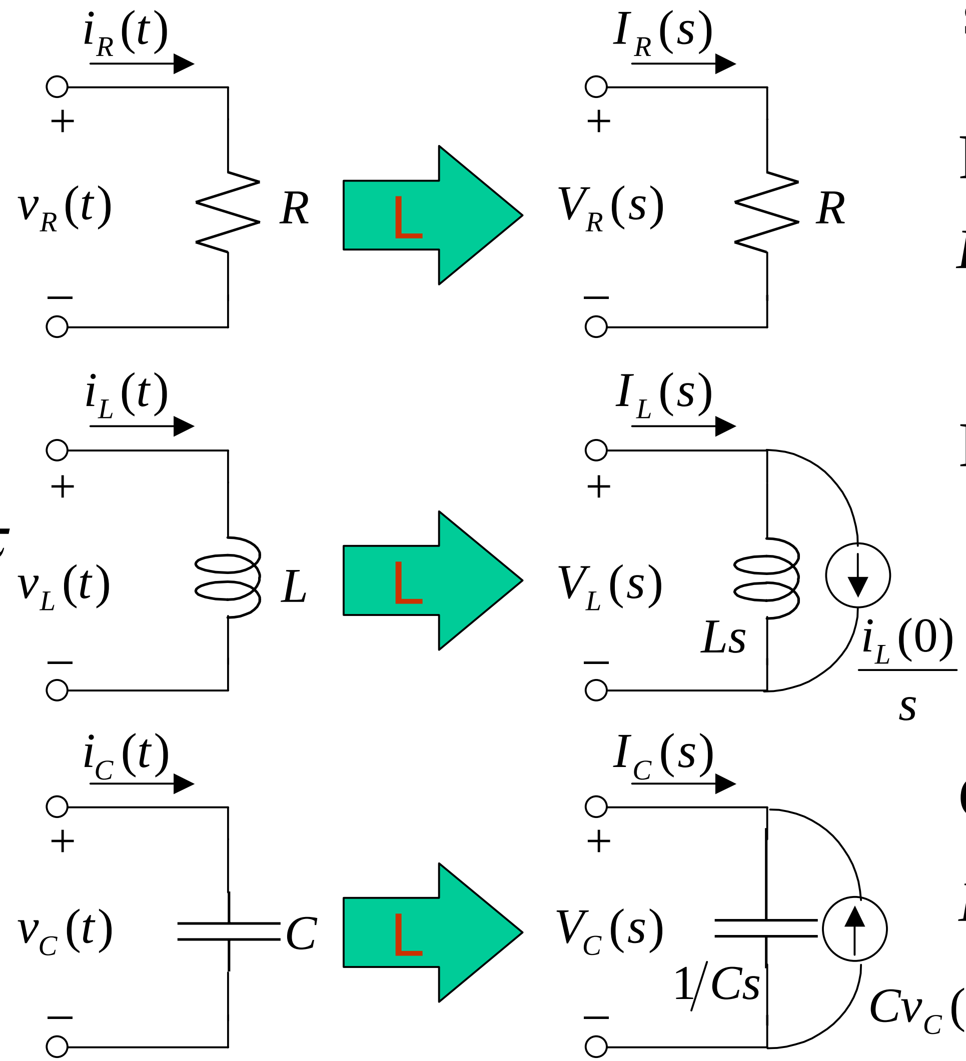
$$i_R(t) = \frac{1}{R} v_R(t)$$

Inductor:

$$i_L(t) = \frac{1}{L}\int_0^t v_L(\tau)d\tau + i_L(0)$$

Capacitor:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$



s-Domain

Resistor:

$$I_R(s) = \frac{1}{R} V_R(s)$$

Inductor:

$$I_L(s) = \frac{1}{Ls} V_L(s) + \frac{i_L(0)}{s}$$

Capacitor:

$$I_C(s) = CsV_C(s) - Cv_C(0)$$

Admittance and Current Source for Initial Conditions

- Admittance Y(s)

$$Y(s) = \frac{\text{current transform}}{\text{voltage transform}} = \frac{1}{Z(s)}$$

with all initial conditions set to zero

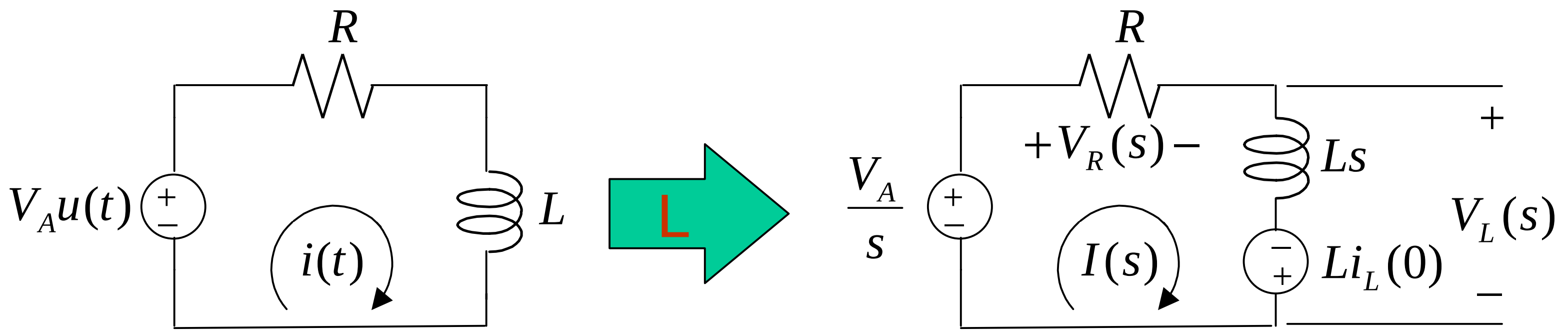
- Admittance of the three passive elements

$$Y_R(s) = \frac{I_R(s)}{V_R(s)} = \frac{1}{R}$$

$$Y_L(s) = \frac{I_L(s)}{V_L(s)} = \frac{1}{Ls} \qquad \text{with } i_L(0) = 0$$

$$Y_C(s) = \frac{I_C(s)}{V_C(s)} = Cs \qquad \text{with } v_C(0) = 0$$

Example: Solve for Current Waveform i(t)



By KVL: $-\dfrac{V_A}{s}+V_R(s)+V_L(s)=0$

Resistor: $V_R(s)=RI(s)$ Inductor: $V_L(s)=LsI(s)-Li_L(0)$

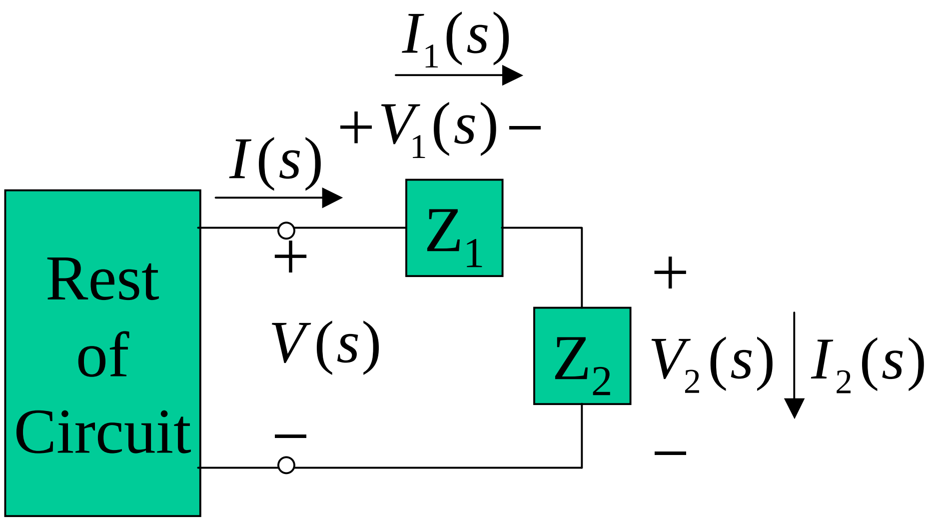
$$-\frac{V_A}{s}+RI(s)+LsI(s)-Li_L(0)=0$$

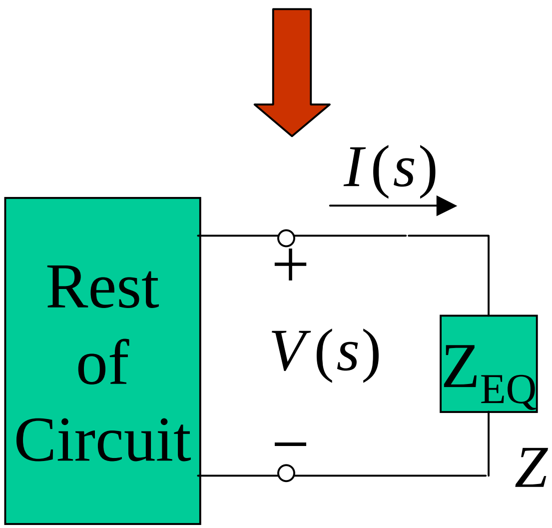
$$I(s)=\frac{V_A/L}{s(s+R/L)}+\frac{i_L(0)}{s+R/L}$$

$$=\frac{V_A/R}{s}-\frac{V_A/R}{s+R/L}+\frac{i_L(0)}{s+R/L}$$

Inverse Transform: $i(t)=\left[\underbrace{\frac{V_A}{R}}_{\text{forced response}}\underbrace{-\frac{V_A}{R}e^{-\frac{R}{L}t}+i_L(0)e^{-\frac{R}{L}t}}_{\text{natural response}}\right]u(t)$

Series Equivalence and Voltage Division





$$Z_{EQ} = Z_1 + Z_2$$

$$V_1(s) = Z_1(s)I_1(s) = Z_1(s)I(s)$$

$$V_2(s) = Z_2(s)I_2(s) = Z_2(s)I(s)$$

$$\text{KVL: } V(s) = V_1(s) + V_2(s)$$

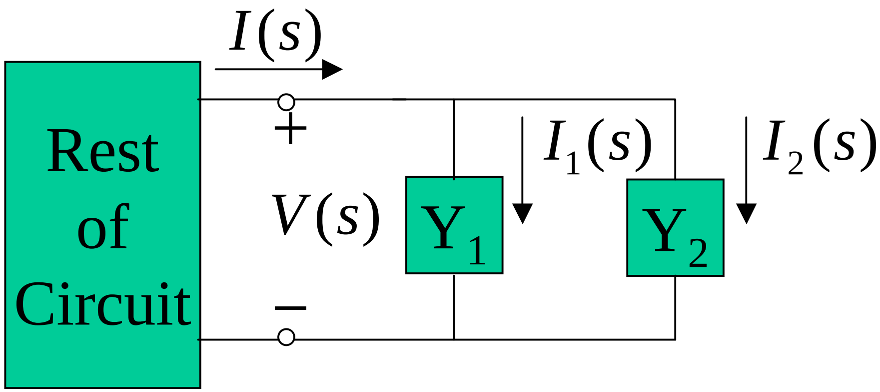
$$= (Z_1(s) + Z_2(s))I(s)$$

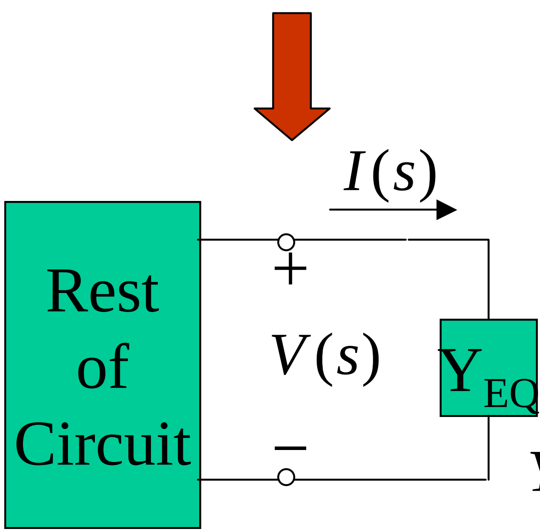
$$\Rightarrow Z_{EQ}(s) = Z_1(s) + Z_2(s)$$

$$V_1(s) = \frac{Z_1(s)}{Z_{EQ}(s)}V(s)$$

$$V_2(s) = \frac{Z_2(s)}{Z_{EQ}(s)}V(s)$$

Parallel Equivalence and Current Division





$$Y_{EQ} = Y_1 + Y_2$$

$$I_1(s) = Y_1(s)V(s)$$

$$I_2(s) = Y_2(s)V(s)$$

$$\text{KCL:}\quad I(s) = I_1(s) + I_2(s)$$

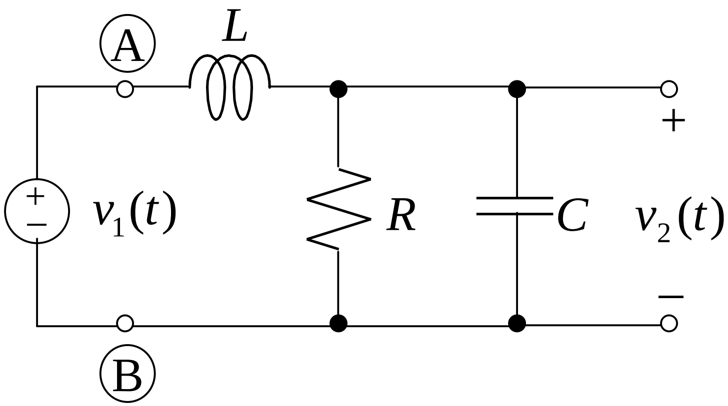
$$= (Y_1(s) + Y_2(s))V(s)$$

$$\Rightarrow \quad Y_{EQ}(s) = Y_1(s) + Y_2(s)$$

$$I_1(s) = \frac{Y_1(s)}{Y_{EQ}(s)} I(s)$$

$$I_2(s) = \frac{Y_2(s)}{Y_{EQ}(s)} I(s)$$

Example:
Equivalence Impedance and Admittance



Inductor current = 0
capacitor voltage = 0
at t = 0

Find equivalent impedance at A and B

Solve for $v_2(t)$

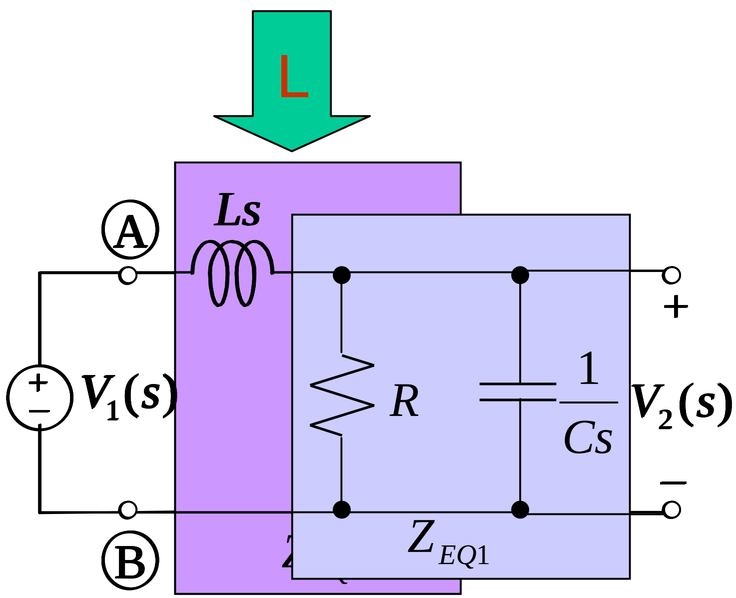
$$Y_{EQ1}(s) = \frac{1}{Z_{EQ1}(s)} = \frac{1}{R} + Cs = \frac{RCs+1}{R}$$

$$Z_{EQ}(s) = Ls + Z_{EQ1}(s) = Ls + \frac{R}{RCs+1}$$

$$= \frac{RLCs^2 + Ls + R}{RCs+1}$$

$$V_2(s) = \frac{Z_{EQ1}(s)}{Z_{EQ}} V_1(s)$$

$$= \frac{R}{RCLs^2 + Ls + R} V_1(s)$$



General Techniques for s-Domain Circuit Analysis

- Node Voltage Analysis (in s-domain)
  - Use Kirchhoff's Current Law (KCL)
  - Get equations of node voltages
  - Use current sources for initial conditions
  - Voltage source $\longrightarrow$ current source

- Mesh Current Analysis (in s-domain)
  - Use Kirchhoff's Voltage Law (KVL)
  - Get equations of currents in the mesh
  - Use voltage sources for initial conditions
  - Current source $\longrightarrow$ voltage source

    (Works only for "Planar" circuits)

Formulating Node-Voltage Equations

Step 0: Transform the circuit into the s domain using current sources to represent capacitor and inductor initial conditions

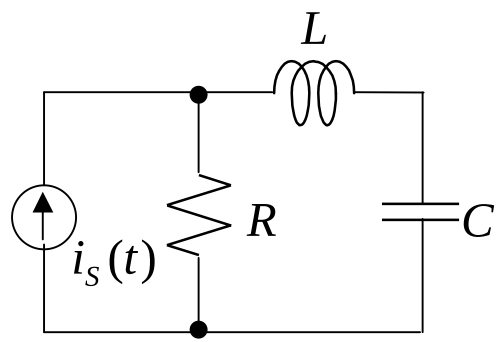
Step 1: Select a reference node. Identify a node voltage at each of the non-reference nodes and a current with every element in the circuit

Step 2: Write KCL connection constraints in terms of the element currents at the non-reference nodes

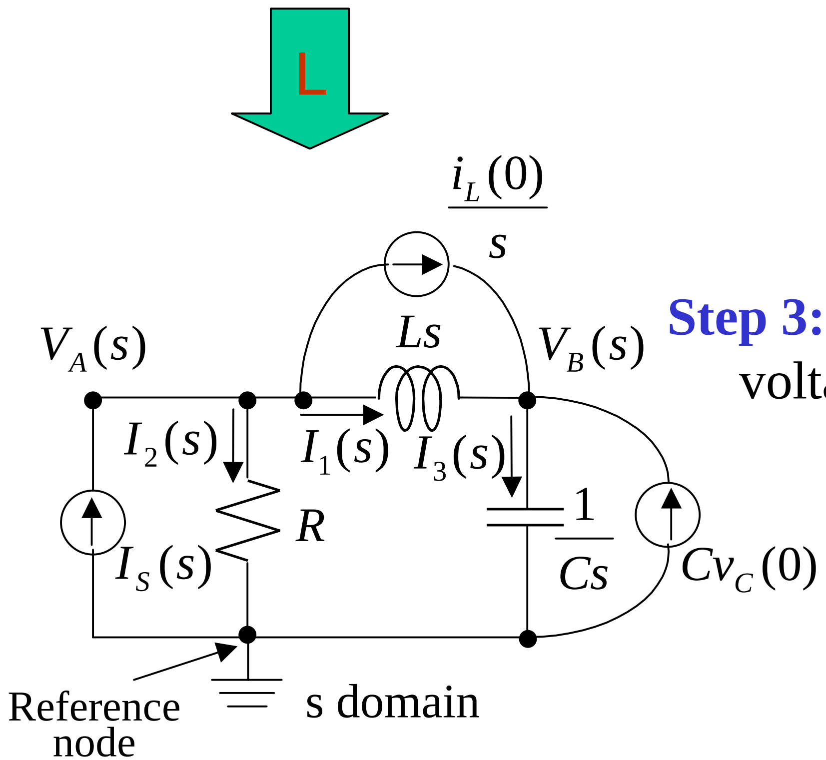
Step 3: Use the element admittances and the fundamental property of node voltages to express the element currents in terms of the node voltages

Step 4: Substitute the device constraints from Step 3 into the KCL connection constraints from Step 2 and arrange the resulting equations in a standard form

Example: Formulating Node-Voltage Equations



t domain



Step 0: Transform the circuit into the s domain using current sources to represent capacitor and inductor initial conditions

Step 1: Identify N-1=2 node voltages and a current with each element

Step 2: Apply KCL at nodes A and B:

$$\text{Node A}: \quad I_S(s) - \frac{i_L(0)}{s} - I_1(s) - I_2(s) = 0$$

$$\text{Node B}: \quad Cv_C(0) + \frac{i_L(0)}{s} + I_1(s) - I_3(s) = 0$$

Step 3: Express element equations in terms of node voltages

$$I_1(s) = Y_L(s)[V_A(s) - V_B(s)] = \frac{1}{Ls}[V_A(s) - V_B(s)]$$

$$I_2(s) = Y_R(s)V_A(s) = GV_A(s) \text{ where } G = 1/R$$

$$I_3(s) = Y_C(s)V_B(s) = CsV_B(s)$$

Formulating Node-Voltage Equations (Cont'd)

Step 2: Apply KCL at nodes A and B:

$$\text{Node A}: \quad I_S(s) - \frac{i_L(0)}{s} - I_1(s) - I_2(s) = 0$$

$$\text{Node B}: \quad Cv_C(0) + \frac{i_L(0)}{s} + I_1(s) - I_3(s) = 0$$

Step 3: Express element equations in terms of node voltages

$$I_1(s) = Y_L(s)\left[V_A(s) - V_B(s)\right] = \frac{1}{Ls}\left[V_A(s) - V_B(s)\right]$$

$$I_2(s) = Y_R(s)V_A(s) = GV_A(s) \text{ where } G = 1/R$$

$$I_3(s) = Y_C(s)V_B(s) = CsV_B(s)$$

Step 4: Substitute eqns. in Step 3 into eqns. in Step 2 and collect common terms to yield node-voltage eqns.

$$\text{Node A}: \quad \left(G + \frac{1}{Ls}\right)V_A(s) - \left(\frac{1}{Ls}\right)V_B(s) = I_S(s) - \frac{i_L(0)}{s}$$

$$\text{Node B}: \quad -\left(\frac{1}{Ls}\right)V_A(s) + \left(\frac{1}{Ls} + Cs\right)V_B(s) = Cv_C(0) + \frac{i_L(0)}{s}$$

Solving s-Domain Circuit Equations

- Circuit Determinant: $\Delta(s) = \begin{vmatrix} G + 1/Ls & -1/Ls \\ -1/Ls & Cs + 1/Ls \end{vmatrix}$

$$= (G + 1/Ls)(Cs + 1/Ls) - (1/Ls)^2$$

$$= \frac{GLCs^2 + Cs + G}{Ls}$$

Depends on circuit element parameters: L, C, G=1/R, not on driving force and initial conditions

- Solve for node A using Cramer's rule:

$$V_A(s) = \frac{\Delta_A(s)}{\Delta(s)} = \frac{\begin{vmatrix} I_S(s) + i_L(0)/s & -1/Ls \\ i_L(0)/s + Cv_C(0) & Cs + 1/Ls \end{vmatrix}}{\Delta(s)}$$

$$= \underbrace{\frac{(LCs^2 + 1)I_S(s)}{GLCs^2 + Cs + G}}_{\substack{\text{Zero State} \\ \text{when initial condition} \\ \text{sources are turned off}}} + \underbrace{\frac{-LCsi_L(0) + Cv_C(0)}{GLCs^2 + Cs + G}}_{\substack{\text{Zero input} \\ \text{when input sources} \\ \text{are turned off}}}$$

Solving s-Domain Circuit Eqns. (Cont'd)

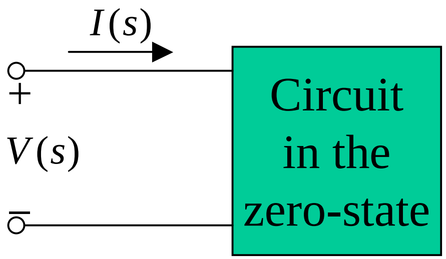
- Solve for node B using Cramer's rule:

$$V_B(s) = \frac{\Delta_B(s)}{\Delta(s)} = \frac{\begin{vmatrix} G + 1/Ls & I_S(s) - i_L(0)/s \\ -1/Ls & i_L(0)/s + Cv_C(0) \end{vmatrix}}{\Delta(s)}$$

$$= \underbrace{\frac{I_S(s)}{GLCs^2 + Cs + G}}_{\text{Zero State}} + \underbrace{\frac{GLi_L(0) + (GLs + 1)Cv_C(0)}{GLCs^2 + Cs + G}}_{\text{Zero input}}$$

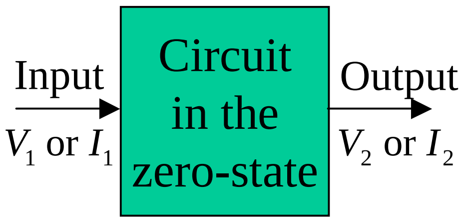
Network Functions

$$\text{Network function} = \frac{\text{Zero-state Response Transform}}{\text{Input Signal Transform}}$$



- Driving-point function relates the voltage and current at a given pair of terminals called a port

$$Z(s) = \frac{V(s)}{I(s)} = \frac{1}{Y(s)}$$



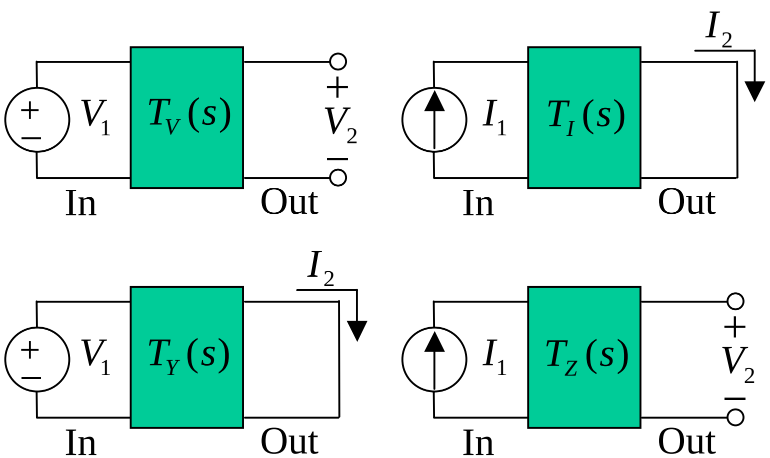
- Transfer function relates an input and response at different ports in the circuit

$$T_V(s) = \text{Voltage Transfer Function} = \frac{V_2(s)}{V_1(s)}$$

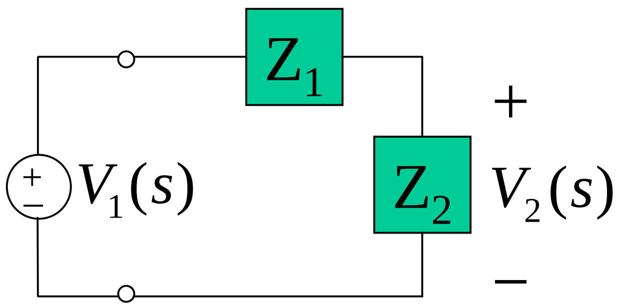
$$T_I(s) = \text{Current Transfer Function} = \frac{I_2(s)}{I_1(s)}$$

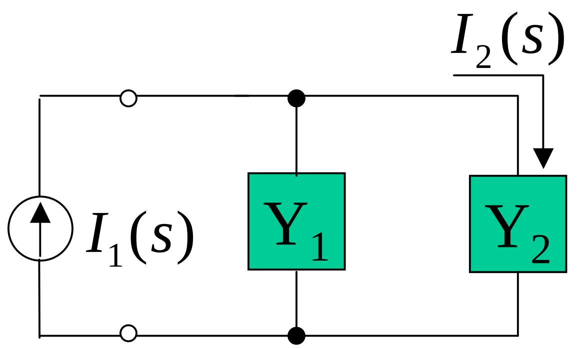
$$T_Y(s) = \text{Transfer Admittance} = \frac{I_2(s)}{V_1(s)}$$

$$T_Z(s) = \text{Transfer Impedance} = \frac{V_2(s)}{I_1(s)}$$



Calculating Network Functions





- Driving-point impedance

$$Z_{EQ}(s) = Z_1(s) + Z_2(s)$$

- Voltage transfer function:

$$V_2(s) = \left[\frac{Z_2(s)}{Z_1(s) + Z_2(s)}\right] V_1(s)$$

$$T_V(s) = \frac{V_2(s)}{V_1(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

- Driving-point admittance

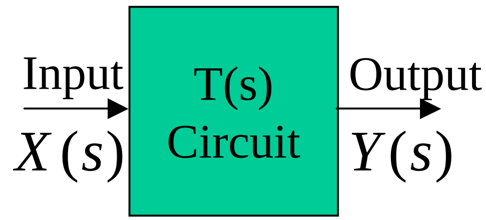
$$Y_{EQ}(s) = Y_1(s) + Y_2(s)$$

- Voltage transfer function:

$$I_2(s) = \left[\frac{Y_2(s)}{Y_1(s) + Y_2(s)}\right] Y_1(s)$$

$$T_I(s) = \frac{I_2(s)}{I_1(s)} = \frac{Y_2(s)}{Y_1(s) + Y_2(s)}$$

Impulse Response and Step Response



- Input-output relationship in s-domain

$$Y(s) = T(s)X(s)$$

- When input signal is an impulse $x(t) = \delta(t)$

$$Y(s) = T(s)\times 1 = T(s)$$

  - Impulse response equals network function
  - H(s) = impulse response transform
  - h(t) = impulse response waveform

- When input signal is a step $x(t) = u(t)$
  - G(s) = step response transform
  - g(t) = step response waveform

$$G(s) = \frac{T(s)}{s} = \frac{H(s)}{s}$$

$$g(s) = \int_0^t h(\tau)d\tau, \quad h(t)(=)\frac{dg(t)}{dt}$$

(=) means equal almost everywhere, excludes those points at which *g(t)* has a discontinuity