Lecture 7

I. Laplace transformation

The Laplace transformation is used when we have some initial conditions in a linear network. The main equations for the Laplace transformations can be derived from the generalised Fourier transformation introduced in Lecture 4. Since the Laplace transformation is applied only along the positive semi-axis $x > 0$, we need just the "+" generalised Fourier transformations – direct and inverse:

$$\hat{f}_+(p) = \int_0^{+\infty} f(x)\exp(-i\,p\,x)dx \qquad (1)$$

$$f(x > 0) = \frac{1}{2\pi}\int_{ia-\infty}^{ia+\infty} \hat{f}_+(p)\exp\left(i\,p\,x\right)dp \qquad (2)$$

where $a < 0$ and $|a|$ is sufficiently large to guarantee the convergence of (1) and (2). Introducing the new variable $ip = s$, we obtain:

$$\hat{f}_+(p) = \breve{f}(s) = \int_0^{+\infty} f(x)\exp(-s\,x)dx \qquad (3)$$

$$f(x > 0) = \frac{1}{2\pi i}\int_{|a|-i\infty}^{|a|+i\infty} \breve{f}(s)\exp\left(s\,x\right)ds \qquad (4)$$

where $\breve{f}(s)$ is the Laplace image and Eq. (4) is the inverse Laplace transformation. For the Laplace transformation, we have the following useful properties:

$$q(t) = \int_0^t g(t-x)h(x)dx \;\Rightarrow\; \breve{q}(s) = \breve{g}(s)\breve{h}(s) \qquad (5)$$

$$\begin{aligned}\int_0^{+\infty} \frac{d^k f(t)}{dt^k}\exp(-st)dt &= s^k\,\breve{f}(s) - \sum_{m=1}^{k} s^{m-1}\frac{d^{k-m} f(0)}{dt^{k-m}} = \\ &= s^n\,\breve{f}(s) - \frac{d^{k-1} f(0)}{dt^{k-1}} - s\frac{d^{k-2} f(0)}{dt^{k-2}} - s^2\frac{d^{k-3} f(0)}{dt^{k-3}} - ... - s^{k-1} f(0)\end{aligned} \qquad (6)$$

Since we will consider only stable networks, the inverse Laplace transformation (4) can be formally expressed through the inverse Fourier transformation:

$$f(x > 0) = \frac{1}{2\pi i}\lim_{|a|\to 0}\int_{|a|-i\infty}^{|a|+i\infty} \breve{f}(s)\exp(sx)ds = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \breve{f}(s)\exp(sx)ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \breve{f}(ip)\exp(ipx)dp \qquad (7)$$

where p is the new integration variable. For the analytical calculation of $\frac{1}{2\pi}\int_{-\infty}^{+\infty}\breve{f}(ip)\exp(ipx)dp$ we can use Eqs. (4.52)–(4.55).

II. Stationary linear networks with initial conditions

In this section, instead of the general variable x we will consider the time variable t. Then, the initial conditions mean that the output parameter $V_{out}(t)$ is subjected to some additional conditions at the time moment $t=0$. (All the explanations will be conducted for a scalar network, but the corresponding generalisations for a vector network can be easily obtained) In turn, the initial conditions by themselves can be represented in the form of some linear (differential, integral, or integro-differential) operators $\mathbf{D}_i$ acting on $V_{out}(t)$: $\mathbf{D}_i V_{out}(t)\big|_{t=0} = C_i$, where i is the integer index and C_i are some real or complex constants. The initial conditions have to define the system behaviour for $t>0$: in compliance with the classical deterministic principle. For example, for a massive material point moving along the real axis, the initial conditions are its position and velocity at $t=0$: $\mathbf{D}_1 x(t)\big|_{t=0} = x(0) = x_0$ and $\mathbf{D}_2 x(t)\big|_{t=0} = \left.\frac{dx(t)}{dt}\right|_{t=0} = v_0$. Actually, the initial conditions can be applied at any time moment, and the time moment $t=0$ has been chosen only for convenience.

If a linear system "is forced" to satisfy the initial conditions, its behaviour or "history" before $t=0$ can be completely ignored when we consider $t>0$. In this case, the convolution integral can be rewritten in the following form:

$$V_{out}(t) = \int_0^t F(t-s)V_{in}(s)ds\,, \tag{8}$$

where t is the current time, $F(t)$ is the kernel of integral operator, $V_{in}(t)$ is the input parameter, and $V_{out}(t)$ is the output parameter. Changing the variable $s=t-q$, the integral in (8) can be rewritten in the following form:

$$V_{out}(t) = \int_0^t F(q)V_{in}(t-q)dq \tag{9}$$

However, for non-zero initial conditions, these convolutions cannot describe all the system features since the integrals will be zero at $t=0$. Therefore, we have to add some additional terms to Eqs. (8) and (9) to satisfy the initial conditions. The corresponding technique will be explained by the example of an ordinary differential equation of the order n with the constant coefficients:

$$\sum_{k=0}^{N} a_k \frac{d^k f(t)}{dt^k} = P\left(\frac{d}{dt}\right) f(t) = A(t) \tag{10}$$

where $P(t)=\sum_{k=0}^{N} a_k t^k$ is the symbol of differential operator (polynomial), $A(t)$ is a function defined only for $t>0$, and a_k are some constant coefficients. This equation can be supplied with the initial conditions:

$$\begin{cases} f(0)=b_0 \\ \left.\dfrac{df(t)}{dt}\right|_{t=0}=b_1 \\ \left.\dfrac{d^2 f(t)}{dt^2}\right|_{t=0}=b_2 \\ . \\ . \\ . \\ \left.\dfrac{d^{N-1} f(t)}{dt^{N-1}}\right|_{t=0}=b_{N-1} \end{cases} \tag{11}$$

where b_i are some constants. Without initial conditions, Eq. (10) can be immediately solved using the operational calculus developed in Lecture 5:

$$f(t)=\int_{-\infty}^{t} E(t-s)A(s)ds \tag{12}$$

where

$$E(t>0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\exp(i\omega t)}{\hat{P}(i\omega)}d\omega=i\sum_{m} res\left[\frac{1}{\hat{P}(iz_m)}\right]\exp(iz_m t) \tag{13}$$

$$\hat{P}(i\omega)=\sum_{k=0}^{N} a_k i^k \omega^k \tag{14}$$

Applying the direct Laplace transformation to Eq. (10) and using the property (6), we obtain:

$$\breve{P}(s)\breve{f}(s)-\sum_{k=1}^{N}\breve{P}_k(s)=\breve{A}(s) \tag{15}$$

where

$$\breve{P}(s)=\sum_{k=0}^{N} a_k s^k \tag{16}$$

$$\breve{P}_k(s)=a_k\sum_{m=1}^{k}\left(s^{m-1}\frac{d^{k-m} f(0)}{dt^{k-m}}\right)=a_k\sum_{m=1}^{k}\left(s^{m-1}b_{k-m}\right) \tag{17}$$

Solving (15) with respect to $\breve{f}(s)$, we obtain:

$$\breve{f}(s)=\frac{\breve{A}(s)}{\breve{P}(s)}+\sum_{k=1}^{N}\frac{\breve{P}_k(s)}{\breve{P}(s)} \tag{18}$$

Applying the inverse Laplace transformation to Eq. (18) and using Eq. (7), we obtain:

$$f(t>0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\breve{A}(i\omega)}{\hat{P}(i\omega)}\exp(i\omega t)d\omega+\sum_{k=1}^{N}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\hat{P}_k(i\omega)}{\hat{P}(i\omega)}\exp(i\omega t)d\omega\right)= \tag{19}$$

Using Eq. (12) with $A(t<0)\equiv 0$, we can rewrite Eq. (19) in the following form:

$$f(t>0)=\int_0^t E(t-s)A(s)ds+\sum_{k=1}^{N}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\hat{P}_k(i\omega)}{\hat{P}(i\omega)}\exp(i\omega t)d\omega\right) \tag{20}$$

Since the order of each polynomial $\hat{P}_k(i\omega)$ is less than the order of $\hat{P}(i\omega)$, the inverse Fourier transformations in Eq. (20) will be classical, and hence it can be calculated using the residues (see (4.53) and (4.54)):

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\hat{P}_k(i\omega)}{\hat{P}(i\omega)}\exp(i\omega t)d\omega&=i\sum_m res\left[\frac{\hat{P}_k(iz_m)}{\hat{P}(iz_m)}\right]\exp(iz_m t)=\\&=i\sum_m\left\{\frac{1}{(n_m-1)!}\lim_{z\to z_m}\left\{\frac{d^{n_m-1}}{dz^{n_m-1}}\left[(z-z_m)^{n_m}\frac{\hat{P}_k(iz)}{\hat{P}(iz)}\right]\right\}\exp(iz_m t)\right\}\end{aligned} \tag{21}$$

Using Eqs. (20),(21), we finally obtain:

$$\begin{aligned}f(t>0)&=\int_0^t E(t-s)A(s)ds+\sum_{k=1}^{N}\left[i\sum_m res\left[\frac{\hat{P}_k(iz_m)}{\hat{P}(iz_m)}\right]\exp(iz_m t)\right]=\\&=\int_0^t E(t-s)A(s)ds+\sum_{k=1}^{N}\left[i\sum_m\left\{\frac{1}{(n_m-1)!}\lim_{z\to z_m}\left\{\frac{d^{n_m-1}}{dz^{n_m-1}}\left[(z-z_m)^{n_m}\frac{\hat{P}_k(iz)}{\hat{P}(iz)}\right]\right\}\exp(iz_m t)\right\}\right]\end{aligned} \tag{22}$$

where

$$\sum_{k=0}^{N}a_k\frac{d^k f(t)}{dt^k}=P\left(\frac{d}{dt}\right)f(t)=A(t) \quad \text{(original equation)}$$

$$b_m=\left.\frac{d^m f(t)}{dt^m}\right|_{t=0} \quad m=0,N-1 \ \ \text{(initial conditions)}$$

$$E(t>0)=i\sum_m res\left[\frac{1}{\hat{P}(iz_m)}\right]\exp(iz_m t)=i\sum_m\left[\frac{1}{(n_m-1)!}\lim_{z\to z_m}\left\{\frac{d^{n_m-1}}{dz^{n_m-1}}\left[\frac{(z-z_m)^{n_m}}{\hat{P}(iz)}\right]\right\}\exp(iz_m t)\right]$$

$$\hat{P}(iz) = \sum_{k=0}^{N} a_k i^k z^k$$

$$\hat{P}_k(iz) = a_k \sum_{m=1}^{k} \left(i^{m-1} z^{m-1} b_{k-m} \right)$$

z_m are zeros of the polynomial $\hat{P}(iz)$

From Eq. (22) we can see that the influence of the initial conditions degrades exponentially with the time. After the certain “characteristic” time, the system will completely forget about its initial conditions and its behaviour will be synchronised with the external stimulus $A(t>0)$. So, the initial conditions are important only during the characteristic time.

We have derived the formula for the Fourier transformation from the ratio of any two polynomials $Q(\omega)$ and $H(\omega)$, if the order of $Q(\omega)$ is less than the order of $H(\omega)$:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{Q(\omega)}{H(\omega)} \exp(i\omega t) d\omega = i \sum_m res\left[\frac{Q(z_m)}{H(z_m)} \right] \exp(iz_m t) = \\ = i \sum_m \left\{ \frac{1}{(n_m - 1)!} \lim_{z \to z_m} \left\{ \frac{d^{n_m - 1}}{dz^{n_m - 1}} \left[(z - z_m)^{n_m} \frac{Q(z)}{H(z)} \right] \right\} \exp(iz_m t) \right\} \end{aligned} \qquad (23)$$

If the order of $Q(\omega)$ is the same or larger than the order of $H(\omega)$, we have to extract the constant or growing part from the ratio, the Fourier transformation from which must be calculated as the generalised one (see (4.53)).

Let us demonstrate the use of Eq. (22):

a) RC circuit

$$\begin{cases} RC \dfrac{dV_C}{dt} + V_C(t) = V_{in}(t) \\ V_C(0) = V_0 \end{cases} \qquad (24)$$

Here, R (resistor), C (capacitor), and $V_{in}(t)$ (generator) are connected in series, V_C is the voltage across the capacitor, and V_0 is the initial voltage. Then:

$\hat{P}(iz) = RCiz + 1$ (has only one zero $z_1 = i/(RC)$)

$\hat{P}_1(iz) = RCV_0 = const$ ($a_1 = RC$ and $b_0 = V_0$)

$$E(t>0) = i \times res\left[\frac{1}{\hat{P}(iz_1)} \right] \exp(iz_1 t) = i \times res\left[\frac{-i}{RC\left(z - \dfrac{i}{RC} \right)} \right] \exp(iz_1 t) = \frac{1}{RC} \exp\left(-\frac{t}{RC} \right)$$

$$i \times res\left[\frac{\hat{P}_1(iz_1)}{\hat{P}(iz_1)}\right]\exp(iz_1 t) = i \times res\left[\frac{-iRCV_0}{RC\left(z - \frac{i}{RC}\right)}\right]\exp(iz_1 t) = V_0 \exp\left(-\frac{t}{RC}\right)$$

$$\boxed{V_C(t>0) = \int_0^t E(t-s)V_{in}(s)ds + i \times res\left[\frac{\hat{P}_k(iz_1)}{\hat{P}(iz_1)}\right]\exp(iz_1 t) = \frac{1}{RC}\int_0^t \exp\left(-\frac{(t-s)}{RC}\right)V_{in}(s)ds + V_0 \exp\left(-\frac{t}{RC}\right) \qquad (25)}$$

For $t >> RC$, the system will completely forget about the initial condition $V_C(0) = V_0$.

b) Damped oscillator

$$\begin{cases} m\frac{d^2x(t)}{dt^2} + g\frac{dx(t)}{dt} + kx(t) = A(t) \\ x(0) = x_0 = b_0 \\ \left.\frac{dx(t)}{dt}\right|_{t=0} = v_0 = b_1 \end{cases} \qquad (26)$$

Here, m is the mass, g is the damping coefficient, k is the Hook coefficient, $A(t)$ is the external force, x_0 is the initial position, and v_0 is the initial velocity. Then:

$$\hat{P}(iz) = -mz^2 + igz + k$$

$$\hat{P}_1(iz) = a_1 b_0 = gx_0$$

$$\hat{P}_2(iz) = a_2 \sum_{m=1}^{2}\left(i^{m-1}z^{m-1}b_{k-m}\right) = a_2 b_1 + a_2 izb_0 = a_2(b_1 + izb_0) = m(v_0 + ix_0 z)$$

$$\begin{cases} \hat{P}(iz) = 0 \quad \Rightarrow -m(z-z_1)(z-z_2) \\ \frac{1}{\hat{P}(iz)} = \frac{-1}{m(z-z_1)(z-z_2)} \\ z_1 = i\frac{g}{2m} + \sqrt{\frac{k}{m} - \frac{g^2}{4m^2}} \\ z_2 = i\frac{g}{2m} - \sqrt{\frac{k}{m} - \frac{g^2}{4m^2}} \end{cases} \qquad \text{(two poles)}$$

$$g < \sqrt{4mk} \text{ is the ringing regime} \Rightarrow \begin{cases} z_1 = i\frac{g}{2m} + \sqrt{\frac{k}{m} - \frac{g^2}{4m^2}} \\ z_2 = i\frac{g}{2m} - \sqrt{\frac{k}{m} - \frac{g^2}{4m^2}} \end{cases} \text{ (complex poles)}$$

$g > \sqrt{4mk}$ is the overdamped regime $\Rightarrow \begin{cases} z_1 = i\left(\dfrac{g}{2m} + \sqrt{\dfrac{g^2}{4m^2} - \dfrac{k}{m}}\right) \\ z_2 = i\left(\dfrac{g}{2m} - \sqrt{\dfrac{g^2}{4m^2} - \dfrac{k}{m}}\right) \end{cases}$ (purely imaginary poles)

It is interesting to note that in the overdamped regime we will have the two different relaxation times: $\tau_1 = \dfrac{1}{\mathrm{Im}[z_1]} = \dfrac{1}{\dfrac{g}{2m} + \sqrt{\dfrac{g^2}{4m^2} - \dfrac{k}{m}}}$ and $\tau_2 = \dfrac{1}{\mathrm{Im}[z_2]} = \dfrac{1}{\dfrac{g}{2m} - \sqrt{\dfrac{g^2}{4m^2} - \dfrac{k}{m}}}$.

Below, we will consider only the ringing regime.

$$E(t>0) = i\sum_m res\left[\frac{1}{\hat{P}(iz_m)}\right]\exp(iz_m t) = i\times res\left[\frac{1}{\hat{P}(iz_1)}\right]\exp(iz_1 t) + i\times res\left[\frac{1}{\hat{P}(iz_2)}\right]\exp(iz_2 t) =$$

$$= i\times res\left[\frac{-1}{m(z-z_1)(z-z_2)}\right]_{z=z_1}\exp(iz_1 t) + i\times res\left[\frac{-1}{m(z-z_1)(z-z_2)}\right]_{z=z_2}\exp(iz_2 t) =$$

$$= -\frac{i\exp(iz_1 t)}{m(z_1-z_2)} - \frac{i\exp(iz_2 t)}{m(z_2-z_1)} = \frac{i\exp\left(-\dfrac{g}{2m}t - it\sqrt{\dfrac{k}{m} - \dfrac{g^2}{4m^2}}\right)}{\sqrt{4mk-g^2}} - \frac{i\exp\left(-\dfrac{g}{2m}t + it\sqrt{\dfrac{k}{m} - \dfrac{g^2}{4m^2}}\right)}{\sqrt{4mk-g^2}} =$$

$$= \frac{2\exp\left(-\dfrac{g}{2m}t\right)\sin\left(t\sqrt{\dfrac{k}{m} - \dfrac{g^2}{4m^2}}\right)}{\sqrt{4mk-g^2}}$$

$$E(t>0) = \frac{2\exp\left(-\dfrac{g}{2m}t\right)\sin\left(t\sqrt{\dfrac{k}{m} - \dfrac{g^2}{4m^2}}\right)}{\sqrt{4mk-g^2}}$$

$$i\sum_m res\left[\frac{\hat{P}_1(iz_m)}{\hat{P}(iz_m)}\right]\exp(iz_m t) = i\times res\left[\frac{\hat{P}_1(iz_1)}{\hat{P}(iz_1)}\right]\exp(iz_1 t) + i\times res\left[\frac{\hat{P}_1(iz_2)}{\hat{P}(iz_2)}\right]\exp(iz_2 t) =$$

$$= i\times res\left[\frac{gx_0}{\hat{P}(iz_1)}\right]\exp(iz_1 t) + i\times res\left[\frac{gx_0}{\hat{P}(iz_2)}\right]\exp(iz_2 t) = \frac{2gx_0\exp\left(-\dfrac{g}{2m}t\right)\sin\left(t\sqrt{\dfrac{k}{m} - \dfrac{g^2}{4m^2}}\right)}{\sqrt{4mk-g^2}}$$

$$i\sum_m res\left[\frac{\hat{P}_1(iz_m)}{\hat{P}(iz_m)}\right]\exp(iz_m t) = \frac{2gx_0\exp\left(-\dfrac{g}{2m}t\right)\sin\left(t\sqrt{\dfrac{k}{m} - \dfrac{g^2}{4m^2}}\right)}{\sqrt{4mk-g^2}}$$

$$i\sum_m res\left[\frac{\hat{P}_2(iz_m)}{\hat{P}(iz_m)}\right]\exp(iz_m t)=i\times res\left[\frac{\hat{P}_2(iz_1)}{\hat{P}(iz_1)}\right]\exp(iz_1 t)+i\times res\left[\frac{\hat{P}_2(iz_2)}{\hat{P}(iz_2)}\right]\exp(iz_2 t)=$$

$$=i\times res\left[\frac{m(v_0+ix_0z_1)}{\hat{P}(iz_1)}\right]\exp(iz_1 t)+i\times res\left[\frac{m(v_0+ix_0z_2)}{\hat{P}(iz_2)}\right]\exp(iz_2 t)=$$

$$=-i\times res\left[\frac{v_0+ix_0z}{(z-z_1)(z-z_2)}\right]_{z=z_1}\exp(iz_1 t)-i\times res\left[\frac{v_0+ix_0z}{(z-z_1)(z-z_2)}\right]_{z=z_2}\exp(iz_2 t)=$$

$$=\frac{i\left(v_0-\frac{gx_0}{2m}\right)+x_0\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}}{2\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}}\exp(iz_2 t)-\frac{i\left(v_0-\frac{gx_0}{2m}\right)-x_0\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}}{2\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}}\exp(iz_1 t)=$$

$$=\frac{2\left(v_0-\frac{gx_0}{2m}\right)\exp\left(-\frac{g}{2m}t\right)\sin\left(t\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}\right)+2x_0\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}\exp\left(-\frac{g}{2m}t\right)\cos\left(t\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}\right)}{2\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}}=$$

$$=\frac{2mv_0-gx_0}{\sqrt{4mk-g^2}}\exp\left(-\frac{g}{2m}t\right)\sin\left(t\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}\right)+x_0\exp\left(-\frac{g}{2m}t\right)\cos\left(t\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}\right)$$

$$i\sum_m res\left[\frac{\hat{P}_2(iz_m)}{\hat{P}(iz_m)}\right]\exp(iz_m t)=\frac{2mv_0-gx_0}{\sqrt{4mk-g^2}}\exp\left(-\frac{g}{2m}t\right)\sin\left(t\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}\right)+x_0\exp\left(-\frac{g}{2m}t\right)\cos\left(t\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}\right)$$

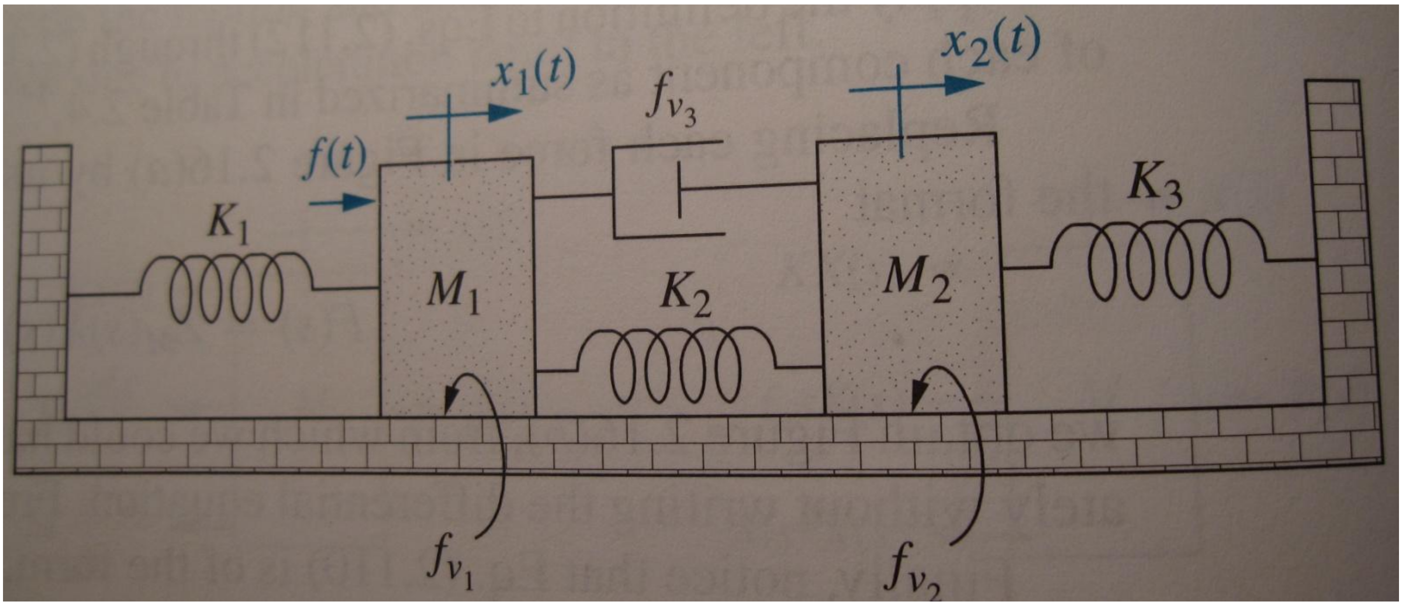
Finally, we obtain for the ringing regime:

$$\begin{aligned}x(t)=&\frac{2}{\sqrt{4mk-g^2}}\int_0^t\exp\left(-\frac{g}{2m}(t-s)\right)\sin\left((t-s)\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}\right)A(s)ds+\\&+x_0\exp\left(-\frac{g}{2m}t\right)\cos\left(t\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}\right)+\frac{(gx_0+2mv_0)}{\sqrt{4mk-g^2}}\exp\left(-\frac{g}{2m}t\right)\sin\left(t\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}\right)\end{aligned}\qquad(27)$$

Using the formula $\frac{d}{dt}\left[\int_{a(t)}^{b(t)}f(s,t)ds\right]=\int_{a(t)}^{b(t)}\frac{df(s,t)}{dt}ds+\frac{db(t)}{dt}f\left(b(t),t\right)-\frac{da(t)}{dt}f\left(a(t),t\right)$, it is easy to check that $x(t)$ satisfies the initial conditions. For $t>>\frac{g}{2m}$, the solution will be synchronised with the external force:

$$x(t)\approx\frac{2}{\sqrt{4mk-g^2}}\int_0^t\exp\left(-\frac{g}{2m}(t-s)\right)\sin\left((t-s)\sqrt{\frac{k}{m}-\frac{g^2}{4m^2}}\right)A(s)ds$$

c) Linear mechanical system with two degrees of freedom (vector linear network)



Here, $f_{v1,2,3}$ are the damping parameters (due to a friction or viscosity), $K_{1,2}$ are the Hook coefficients (springs), and $M_{1,2}$ are the masses. In accordance with Newton's law:

$$\begin{cases} M_1 \dfrac{d^2 x_1(t)}{dt^2} = f(t) - K_1 x_1(t) + K_2\left(x_2(t) - x_1(t)\right) + f_{v3}\left(\dfrac{dx_2(t)}{dt} - \dfrac{dx_1(t)}{dt}\right) - f_{v1}\dfrac{dx_1(t)}{dt} \\ M_2 \dfrac{d^2 x_2(t)}{dt^2} = -K_3 x_2(t) + K_2\left(x_1(t) - x_2(t)\right) + f_{v3}\left(\dfrac{dx_1(t)}{dt} - \dfrac{dx_2(t)}{dt}\right) - f_{v2}\dfrac{dx_2(t)}{dt} \end{cases} \qquad (28)$$

In Eq. (28), all the forces are grouped at the right to demonstrate their physical meaning:

$\begin{cases} -K_1 x_1(t) \\ -K_3 x_2(t) \end{cases}$ These "self" Hook's forces will be opposite to $f(t)$, if the direction of displacements $x_{1,2}(t)$ coincides with the direction of $f(t)$. This is an obvious physical fact.

$\begin{cases} K_2\left(x_2(t) - x_1(t)\right) \\ K_2\left(x_1(t) - x_2(t)\right) \end{cases}$ These "mutual" Hook's forces depend on the relative displacement $\left(x_2(t) - x_1(t)\right)$ or $\left(x_1(t) - x_2(t)\right)$.

To define the proper signs for these forces, let us consider the situation when both the displacements $x_{1,2}(t)$ have the same direction as $f(t)$, but $x_2(t) > x_1(t)$. In this case, the spring K_2 will pull forward the mass M_1 ($x_2(t) - x_1(t) > 0$) and back the mass M_2 ($x_1(t) - x_2(t) < 0$). Here, "forward" means in the direction of $f(t)$, and "back" means in the opposite direction.

$$\begin{cases} f_{v3}\left(\dfrac{dx_2(t)}{dt}-\dfrac{dx_1(t)}{dt}\right) \\ f_{v3}\left(\dfrac{dx_1(t)}{dt}-\dfrac{dx_2(t)}{dt}\right) \end{cases}$$

These "mutual" friction forces depend on the relative velocity $\left(\dfrac{dx_2(t)}{dt}-\dfrac{dx_1(t)}{dt}\right)$ or $\left(\dfrac{dx_1(t)}{dt}-\dfrac{dx_2(t)}{dt}\right)$.

To define the proper signs for these forces, let us consider the situation when both the velocities $\dfrac{dx_{1,2}(t)}{dt}$ have the same direction as $f(t)$, but $\dfrac{dx_2(t)}{dt} > \dfrac{dx_1(t)}{dt}$. In this case, the damper f_{v3} will pull forward the mass M_1 $\left(\dfrac{dx_2(t)}{dt}-\dfrac{dx_1(t)}{dt}>0\right)$ and back the mass M_2 $\left(\dfrac{dx_1(t)}{dt}-\dfrac{dx_2(t)}{dt}<0\right)$ Here, "forward" means in the direction of $f(t)$, and "back" means in the opposite direction.

$$\begin{cases} -f_{v1}\dfrac{dx_1(t)}{dt} \\ -f_{v2}\dfrac{dx_2(t)}{dt} \end{cases}$$

These "self" friction forces will be opposite to $f(t)$, if the direction of velocities $\dfrac{dx_{1,2}(t)}{dt}$ coincides with the direction of $f(t)$. This is an obvious physical fact.

The system (28) can be rewritten in the following equivalent form:

$$\begin{cases} M_1\dfrac{d^2x_1(t)}{dt^2}+(f_{v1}+f_{v3})\dfrac{dx_1(t)}{dt}+(K_1+K_2)x_1(t)-f_{v3}\dfrac{dx_2(t)}{dt}-K_2x_2(t)=f(t) \\ M_2\dfrac{d^2x_2(t)}{dt^2}+(f_{v2}+f_{v3})\dfrac{dx_2(t)}{dt}+(K_3+K_2)x_2(t)-f_{v3}\dfrac{dx_1(t)}{dt}-K_2x_1(t)=0 \end{cases}$$

It can be supplied with the initial conditions:

$$\begin{cases} x_1(0)=x_{10} \\ \left.\dfrac{dx_1(t)}{dt}\right|_{t=0}=v_{10} \\ x_2(0)=x_{20} \\ \left.\dfrac{dx_2(t)}{dt}\right|_{t=0}=v_{20} \end{cases}$$

Applying the direct Laplace transformation (3) and using Eq. (6), we obtain:

$$\begin{cases} \left[M_1s^2+(f_{v1}+f_{v3})s+(K_1+K_2)\right]\breve{x}_1(s)-(f_{v1}+f_{v3})x_{10}-M_1(v_{10}+x_{10}s)-[f_{v3}s+K_2]\breve{x}_2(s)+f_{v3}x_{20}=\breve{f}(s) \\ \left[M_2s^2+(f_{v2}+f_{v3})s+(K_3+K_2)\right]\breve{x}_2(s)-(f_{v2}+f_{v3})x_{20}-M_2(v_{20}+x_{20}s)-[f_{v3}s+K_2]\breve{x}_1(s)+f_{v3}x_{10}=0 \end{cases}$$

$$\begin{cases} \left[M_1 s^2 + \left(f_{v1} + f_{v3}\right)s + \left(K_1 + K_2\right)\right]\breve{x}_1(s) - \left[f_{v3}s + K_2\right]\breve{x}_2(s) = \breve{f}(s) + M_1 x_{10}s + \left(f_{v1} + f_{v3}\right)x_{10} - f_{v3}x_{20} + M_1 v_{10} \\ \\ -\left[f_{v3}s + K_2\right]\breve{x}_1(s) + \left[M_2 s^2 + \left(f_{v2} + f_{v3}\right)s + \left(K_3 + K_2\right)\right]\breve{x}_2(s) = M_2 x_{20}s + \left(f_{v2} + f_{v3}\right)x_{20} - f_{v3}x_{10} + M_2 v_{20} \end{cases}$$

For this system of algebraic equations, we can introduce the following polynomials of the variable s:

$$\begin{cases} \breve{A}(s) = M_1 s^2 + \left(f_{v1} + f_{v3}\right)s + \left(K_1 + K_2\right) \\ \breve{B}(s) = M_2 s^2 + \left(f_{v2} + f_{v3}\right)s + \left(K_3 + K_2\right) \\ \breve{C}(s) = f_{v3}s + K_2 \\ \\ \breve{D}_1(s) = M_1 x_{10}s + \left(f_{v1} + f_{v3}\right)x_{10} - f_{v3}x_{20} + M_1 v_{10} \\ \breve{D}_2(s) = M_2 x_{20}s + \left(f_{v2} + f_{v3}\right)x_{20} - f_{v3}x_{10} + M_2 v_{20} \end{cases}$$

Then:

$$\begin{cases} \breve{A}(s)\breve{x}_1(s) - \breve{C}(s)\breve{x}_2(s) = \breve{f}(s) + D_1(s) \\ \\ -\breve{C}(s)\breve{x}_1(s) + \breve{B}(s)\breve{x}_2(s) = D_2(s) \end{cases}$$

This system can be easily solved:

$$\breve{x}_1(s) = \breve{f}(s)\frac{\breve{B}(s)}{\breve{A}(s)\breve{B}(s) - \breve{C}^2(s)} + \frac{\breve{B}(s)\breve{D}_1(s)}{\breve{A}(s)\breve{B}(s) - \breve{C}^2(s)} + \frac{\breve{C}(s)\breve{D}_2(s)}{\breve{A}(s)\breve{B}(s) - \breve{C}^2(s)}$$

$$\breve{x}_2(s) = \breve{f}(s)\frac{\breve{C}(s)}{\breve{A}(s)\breve{B}(s) - \breve{C}^2(s)} + \frac{\breve{C}(s)\breve{D}_1(s)}{\breve{A}(s)\breve{B}(s) - \breve{C}^2(s)} + \frac{\breve{A}(s)\breve{D}_2(s)}{\breve{A}(s)\breve{B}(s) - \breve{C}^2(s)}$$

Similar to Eq. (13), we can introduce the following time domain transfer functions used in the convolutions with the external force $f(t)$:

$$\begin{aligned} E_1(t>0) &= i\sum_m res\left[\frac{\hat{B}(iz_m)}{\hat{A}(iz_m)\hat{B}(iz_m) - \hat{C}^2(iz_m)}\right]\exp(iz_m t) = \\ &= i\sum_m \left[\frac{1}{(n_m-1)!}\lim_{z\to z_m}\left\{\frac{d^{n_m-1}}{dz^{n_m-1}}\left[\frac{\hat{B}(iz)(z-z_m)^{n_m}}{\hat{A}(iz)\hat{B}(iz) - \hat{C}^2(iz)}\right]\right\}\exp(iz_m t)\right] \end{aligned}$$

$$\begin{aligned} E_2(t>0) &= i\sum_m res\left[\frac{\hat{C}(iz_m)}{\hat{A}(iz_m)\hat{B}(iz_m) - \hat{C}^2(iz_m)}\right]\exp(iz_m t) = \\ &= i\sum_m \left[\frac{1}{(n_m-1)!}\lim_{z\to z_m}\left\{\frac{d^{n_m-1}}{dz^{n_m-1}}\left[\frac{\hat{C}(iz)(z-z_m)^{n_m}}{\hat{A}(iz)\hat{B}(iz) - \hat{C}^2(iz)}\right]\right\}\exp(iz_m t)\right] \end{aligned}$$

where z_m are zeros of the denominator $\Delta = \hat{A}(iz)\hat{B}(iz) - \hat{C}^2(iz)$. Finally, we obtain:

$$x_1(t>0) = \int_0^t E_1(t-s)f(s)ds +$$
$$+i\sum_m res\left[\frac{\hat{B}(iz_m)\hat{D}_1(iz_m)}{\hat{A}(iz_m)\hat{B}(iz_m) - \hat{C}^2(iz_m)}\right]\exp(iz_m t) + i\sum_m res\left[\frac{\hat{C}(iz_m)\hat{D}_2(iz_m)}{\hat{A}(iz_m)\hat{B}(iz_m) - \hat{C}^2(iz_m)}\right]\exp(iz_m t)$$

(29)

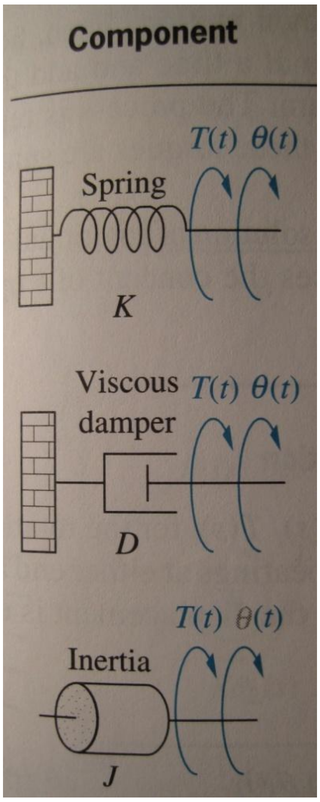
$$x_2(t>0) = \int_0^t E_2(t-s)f(s)ds +$$
$$+i\sum_m res\left[\frac{\hat{C}(iz_m)\hat{D}_1(iz_m)}{\hat{A}(iz_m)\hat{B}(iz_m) - \hat{C}^2(iz_m)}\right]\exp(iz_m t) + i\sum_m res\left[\frac{\hat{A}(iz_m)\hat{D}_2(iz_m)}{\hat{A}(iz_m)\hat{B}(iz_m) - \hat{C}^2(iz_m)}\right]\exp(iz_m t)$$

These solutions are very complicated, since we have to find zeros of $\Delta = \hat{A}(iz)\hat{B}(iz) - \hat{C}^2(iz)$. However, it is still possible to do analytically (!) since the order of this polynomial is 4. If the initial conditions are zero, we obtain:

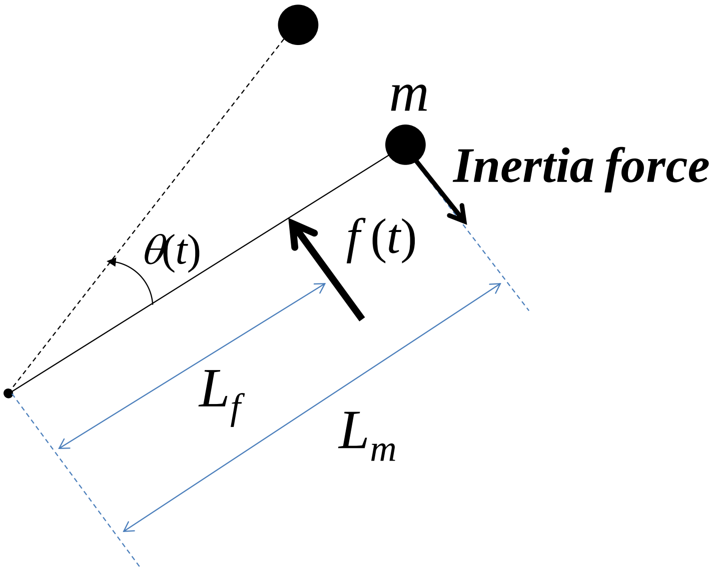
$$x_1(t>0) = \int_0^t E_1(t-s)f(s)ds$$

$$x_2(t>0) = \int_0^t E_2(t-s)f(s)ds$$

III. Rotational mechanical systems



The following simple model (see the Figure below) will help us to derive the main equations for a rotational solid. The mass m is fastened on the arm L_m which can rotate around the centre. The external force $f(t)$ is applied at right angle to this arm at some distance L_f (L_f may be larger or smaller than L_m):



For the external and inertia forces, we can write the moment equality for each moment of time t:

$$L_f f(t) = L_m m a(t) \tag{30}$$

Here, $v(t) = L_m \dfrac{d\theta(t)}{dt} = L_m \omega(t)$ is the linear velocity, $\omega(t)$ is the angular velocity, and $a(t) = \dfrac{dv(t)}{dt} = L_m \dfrac{d^2\theta(t)}{dt^2} = L_m \dfrac{d\omega(t)}{dt}$ is the angular acceleration. Therefore, Eq. (30) can be rewritten in the following form:

$$L_f f(t) = L_m m \left(L_m \frac{d^2\theta(t)}{dt^2} \right) = L_m^2 m \frac{d^2\theta(t)}{dt^2} \tag{31}$$

By the definition, $L_f f(t)$ is the scalar torque $T(t)$ (Newton×Meter; SI units) and $L_m^2 m$ (Mass×Meter^2; SI utins) is the moment of inertia J (constant!). Finally, we obtain:

$$T(t) = J \frac{d^2\theta(t)}{dt^2} = J \frac{d\omega(t)}{dt} \tag{32}$$

The mechanical power $W(t)$ (Joule/Second or Watt; SI units) is calculated as the work produced by the external force per the unit time. Using our simple mechanical model, we can use the infinitesimal analysis to calculate $W(t)$:

$W(t) = \lim\limits_{\Delta t \to 0} \dfrac{L_f \left[f(t+\Delta t)\theta(t+\Delta t) - f(t)\theta(t) \right]}{\Delta t}$ is the mechanical power.

Then, using Taylor's expansions we obtain:

$$W(t)=\lim_{\Delta t\to 0}\frac{L_f\left[f(t+\Delta t)\theta(t+\Delta t)-f(t)\theta(t)\right]}{\Delta t}=\lim_{\Delta t\to 0}\frac{L_f\left[\left(f(t)+\frac{df(t)}{dt}\Delta t\right)\left(\theta(t)+\frac{d\theta(t)}{dt}\Delta t\right)-f(t)\theta(t)\right]}{\Delta t}=$$
$$=L_f\left[\frac{df(t)}{dt}\theta(t)+\frac{d\theta(t)}{dt}f(t)\right]=\frac{dT(t)}{dt}\theta(t)+\frac{d\theta(t)}{dt}T(t)=\frac{dT(t)}{dt}\theta(t)+\omega(t)T(t)$$

$$W(t)=\frac{dT(t)}{dt}\theta(t)+\omega(t)T(t)=\frac{d}{dt}\left(T(t)\theta(t)\right) \tag{33}$$

If $T(t)=const$, then $W(t)=\omega(t)T$.

$A=\int_{t_1}^{t_2}W(t)dt=\int_{t_1}^{t_2}\frac{d}{dt}\left(T(t)\theta(t)\right)dt=T(t_2)\theta(t_2)-T(t_1)\theta(t_1)=\Delta\left[T(t)\theta(t)\right]$ is the mechanical work produced during the time interval $[t_1,t_2]$.

In the general case, the mass and the external force may be distributed (non-point force and mass), but Eqs. (32),(33) remain the same. For the distributed mass and external force, $T(t)$ and J must be calculated with respect to the rotation axis:

$$J=\int_V\rho(x,y,z)R^2(x,y,z)dV \tag{34}$$

$$T(t)=\int_V\left([\mathbf{n}\times\mathbf{R}(x,y,z)],\mathbf{f}(t;x,y,z)\right)dV \tag{35}$$

Here, $dV=dxdydz$ is the elementary volume, $\rho(x,y,z)$ is the mass volume density, V is the volume of solid, $\mathbf{n}$ is the unit vector directed along the rotation axis, $\mathbf{R}(x,y,z)$ is the radius-vector which is perpendicular to the rotation axis and directed from this axis to the integration point inside V, where the vector force $\mathbf{f}(t;x,y,z)$ is applied, $R(x,y,z)=|\mathbf{R}(x,y,z)|$ is the module of $\mathbf{R}(x,y,z)$, [_×_] is the vector product of two vectors, (_,_) is the scalar product of two vector. In (35), we have the mixed product of three vectors. Eq. (35) shows that a vector force $\mathbf{f}(t;x,y,z)$ applied to the solid produces a torque only if it has a component along the vector $[\mathbf{n}\times\mathbf{R}(x,y,z)]$, i.e. perpendicular to the rotation plane. All other components will not contribute to the torque.

For Eq. (32), we can also introduce the additional frictional $D\frac{d\theta(t)}{dt}$ and elastic $K\theta(t)$ torques (they have the same dimension as T):

$$J\frac{d^2\theta(t)}{dt^2}+D\frac{d\theta(t)}{dt}+K\theta(t)=T(t) \tag{36}$$

Eq. (36) can be supplied with the initial conditions:

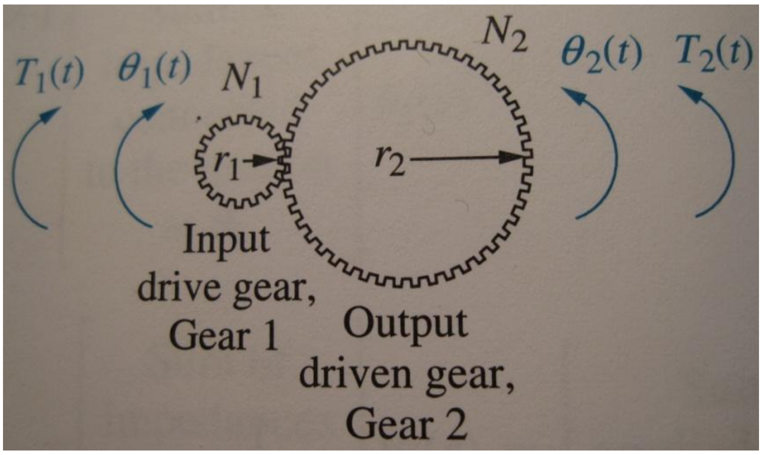
$$\begin{cases} \theta(0) = \theta_0 \\ \left.\dfrac{d\theta(t)}{dt}\right|_{t=0} = \omega(0) = \omega_0 \end{cases} \tag{37}$$

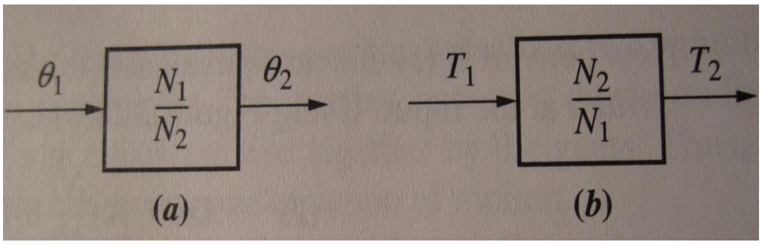
Since Eqs. (35) and (36) are similar to Eq. (26), we can use the same analytical tool.

Gears transform both the speed of rotation and the applied torque. For two gears (1,2) with the radius $r_{1,2}$ and the number of teeth $N_{1,2}$ respectively, we have the following ratios:

$$\begin{cases} \dfrac{\theta_2(t)}{\theta_1(t)} = \dfrac{r_1}{r_2} = \dfrac{N_1}{N_2} \quad \Rightarrow \quad \theta_2(t) = \dfrac{r_1}{r_2}\theta_1(t) = \dfrac{N_1}{N_2}\theta_1(t) \\ \\ \dfrac{T_2(t)}{T_1(t)} = \dfrac{r_2}{r_1} = \dfrac{N_2}{N_1} \quad \Rightarrow \quad T_2(t) = \dfrac{r_2}{r_1}T_1(t) = \dfrac{N_2}{N_1}T_1(t) \end{cases} \tag{38}$$

Therefore, a gear mechanism can be considered as a linear network, as shown in the Figures below.



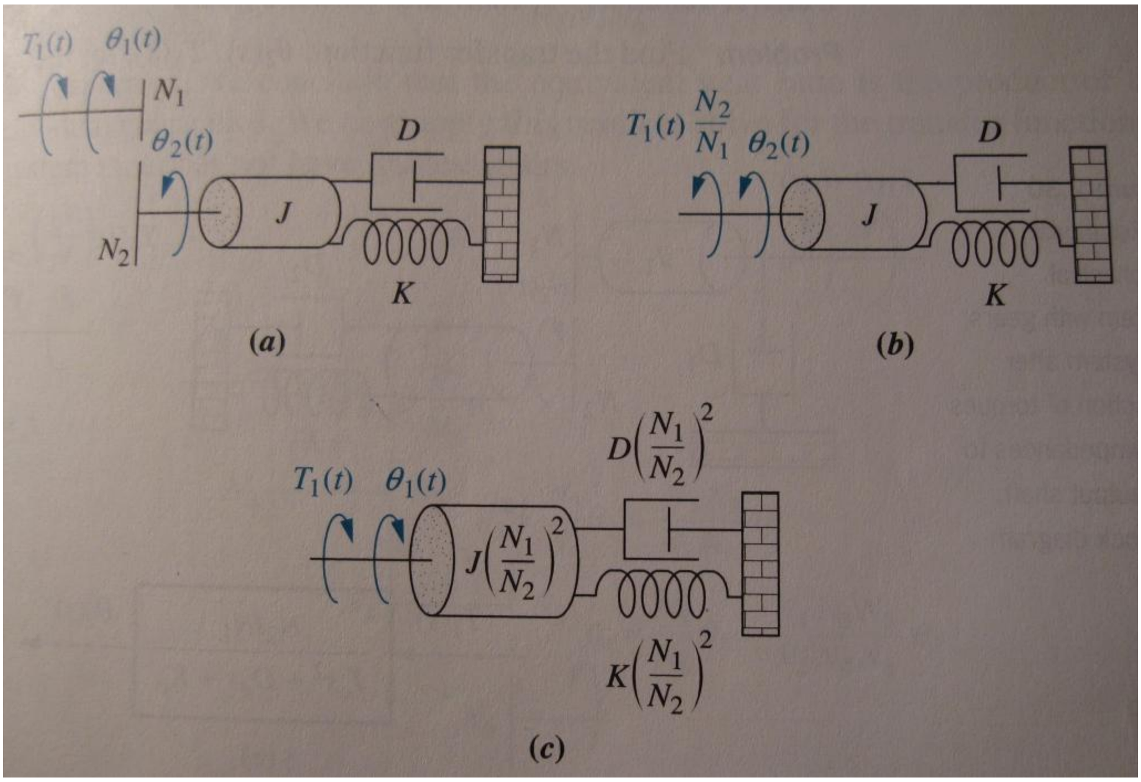


For a rotational solid attached to the second gear, we have (see (a) in the Figure below):

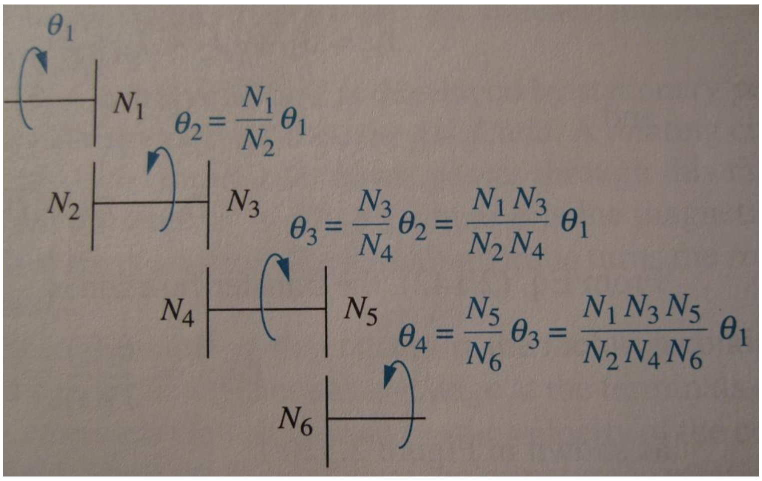
$$J\frac{d^2\theta_2(t)}{dt^2} + D\frac{d\theta_2(t)}{dt} + K\theta_2(t) = T_2(t) \tag{39}$$

Putting $\theta_2(t) = \theta_1(t)\dfrac{N_1}{N_2}$ and $T_2(t) = T_1(t)\dfrac{N_2}{N_1}$ into Eq. (36), we obtain (see (b),(c) in the Figure below):

$$J\left(\frac{N_1}{N_2}\right)^2 \frac{d^2\theta_1(t)}{dt^2} + D\left(\frac{N_1}{N_2}\right)^2 \frac{d\theta_1(t)}{dt} + K\left(\frac{N_1}{N_2}\right)^2 \theta_1(t) = T_1(t) \tag{40}$$

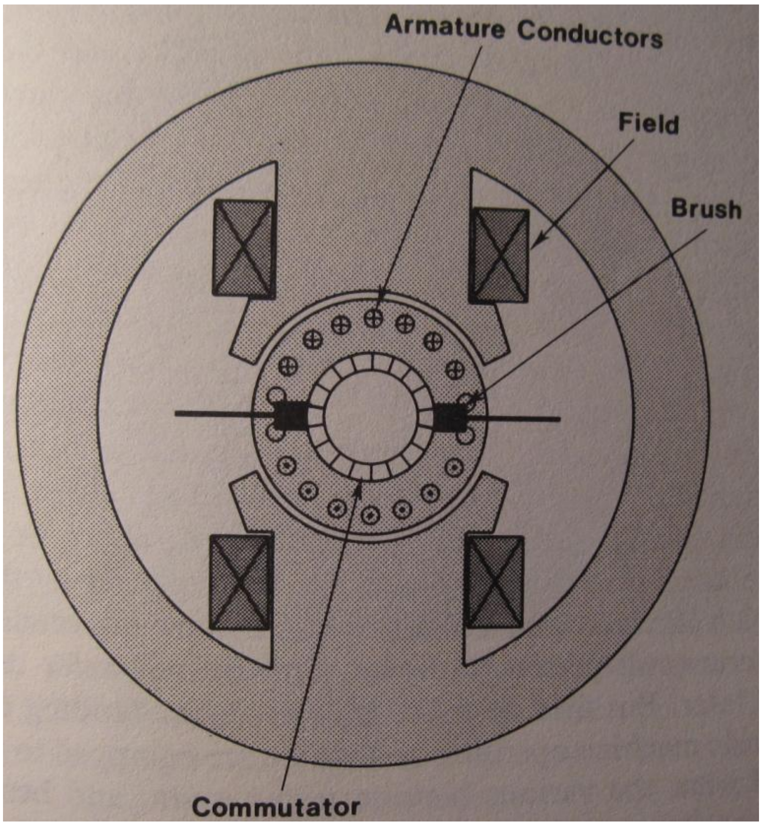


For a gear train, the equivalent gear ratio is the product of the individual gear ratios, as shown in the Figure below.



IV. Electromechanical systems

Any electrical motor together with a mechanical system attached constitute an electromechanical system. For such system, electrical and mechanical properties must be considered together since they affect each other. In this section, we will study the voltage-controlled DC motor. The driving magnetic field in this motor is induced by stationary permanent magnets or electromagnets. This motionless part of the motor is called the stator. The armature coils, which are assembled on the rotor (rotating part), are connected through the commutator and brushes to a voltage source.



The motor torque $T(t)$ is developed due to the interaction of the armature current $I_a(t)$ with the magnetic field induced by the stator:

$$T(t) = K_t I_a(t) \tag{41}$$

where K_t is the proportionality coefficient that depends linearly on the stator magnetic field (which is assumed to be uniform in the stator-to-rotor gap) and also on the design of armature coils. If the stator is made of electromagnets, K_t will depend on the stator current $I_s(t)$:

$$K_t(t) = K_{ts} I_s(t) \tag{42}$$

where K_{ts} is the proportionality constant which is defined by the design of stator and rotor.

The armature and stator coils by themselves are characterised by the inductances L_a and L_s, and the resistances R_a and R_s, respectively. In the armature circuit, we also have to take into account the so-called back e.m.f. $V_b(t)$, which is induced in the rotating armature coils by the stator magnetic field (due to Faraday's law). This back e.m.f. is directly proportional to the speed of rotation:

$$V_b(t) = -K_b \frac{d\theta(t)}{dt} \tag{43}$$

where K_b is the proportionality coefficient that depends linearly on the stator magnetic field (which is assumed to be uniform in the stator-to-rotor gap) and also on the design of armature coils. If the stator is made of electromagnets, K_b will depend on the stator current $I_s(t)$:

$$K_b(t) = K_{bs} I_s(t) \tag{44}$$

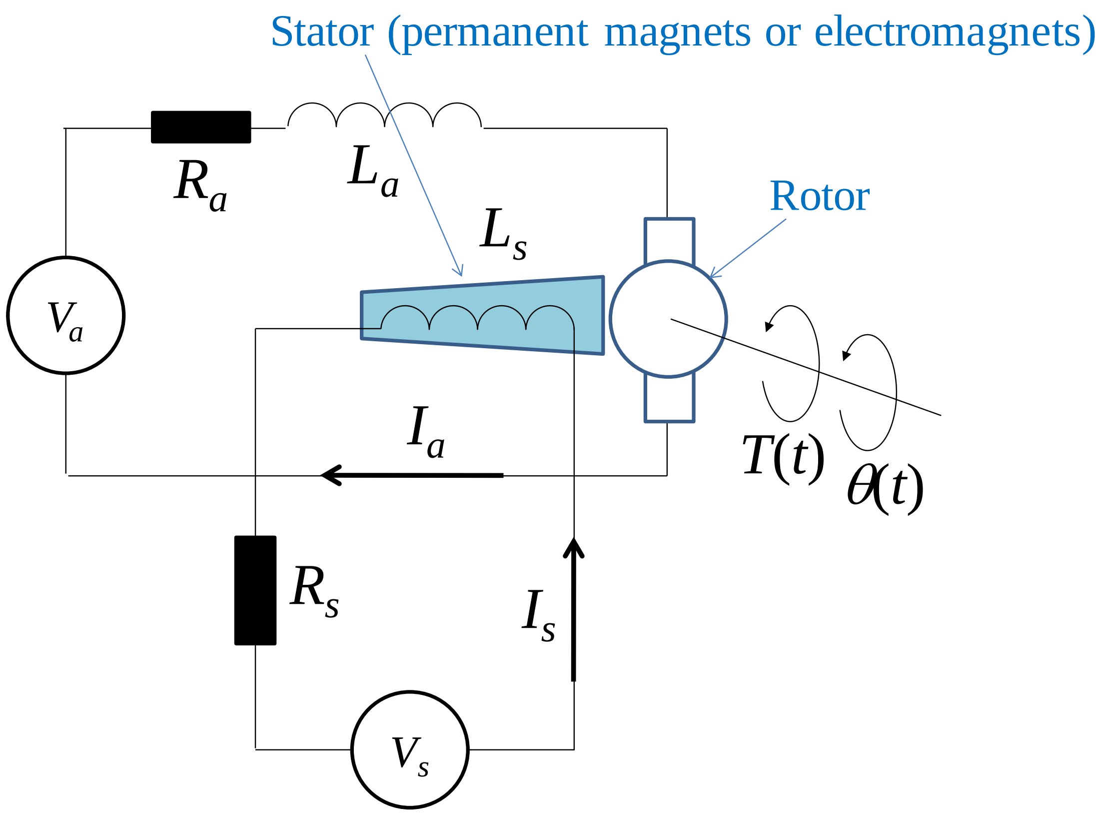
where K_{bs} is the proportionality constant which is defined by the design of stator and rotor.

The simplest feeding scheme is realised when the armature and stator coils are connected to the different voltage sources, as shown in the figure below. In this case, K_t and K_b are constants, and we have the following system of differential equations for the armature current $I_a(t)$ and the rotation angle $\theta(t)$:

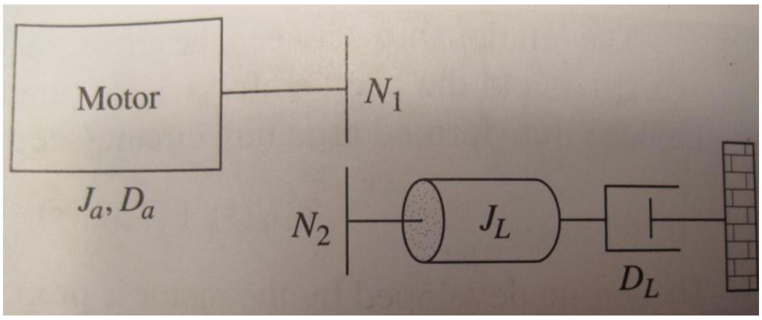
$$R_a I_a(t) + L_a \frac{dI_a(t)}{dt} + K_b \frac{d\theta(t)}{dt} = V_a(t) \tag{45}$$

$$J \frac{d^2\theta(t)}{dt^2} + D \frac{d\theta(t)}{dt} + K\theta(t) = T(t) \tag{46}$$

$$T(t) = K_t I_a(t) \tag{47}$$



We will consider the system (45)–(47) with zero initial conditions. There: $J = J_a + J_L \left(\frac{N_1}{N_2}\right)^2$ is the total moment of inertia, including the armature and the external load, $D = D_a + D_L \left(\frac{N_1}{N_2}\right)^2$ is the total damping parameter, and $K = K_a + K_L \left(\frac{N_1}{N_2}\right)^2$ is the total elastic parameter. Here, we assume that a rotational external load can be connected to the motor through a reduction gear, as shown in the figure below.



Applying the direct Laplace transformation to Eqs. (45)–(47), we obtain (zero initial conditions!):

$$R_a \breve{I}_a(s) + L_a s \breve{I}_a(s) + K_b s \breve{\theta}(s) = \breve{V}_a(s) \tag{48}$$

$$Js^2 \breve{\theta}(s) + Ds \breve{\theta}(s) + K \breve{\theta}(s) = \breve{T}(s) \tag{49}$$

$$\breve{T}(s) = K_t \breve{I}_a(s) \tag{50}$$

Solving Eqs. (48)–(47) with respect to $\breve{I}_a(s)$ and $\breve{\theta}(s)$, we obtain:

$$\breve{I}_a(s) = \breve{V}_a(s) \frac{(Js^2 + Ds + K)}{JL_a s^3 + (JR_a + DL_a)s^2 + (DR_a + L_a K + K_t K_b)s + R_a K} \tag{51}$$

$$\boxed{\breve{\theta}(s) = \breve{V}_a(s) \frac{K_t}{JL_a s^3 + (JR_a + DL_a)s^2 + (DR_a + L_a K + K_t K_b)s + R_a K}} \tag{52}$$

For further analysis, we will neglect K:

$$\breve{I}_a(s) = \breve{V}_a(s) \frac{(Js + D)}{JL_a s^2 + (JR_a + DL_a)s + (DR_a + K_t K_b)} \tag{53}$$

$$\boxed{\breve{\theta}(s) = \breve{V}_a(s) \frac{K_t}{s(JL_a s^2 + (JR_a + DL_a)s + (DR_a + K_t K_b))}} \tag{54}$$

Let us introduce the following two polynomials:

$$P(z) = JL_a z^2 + (JR_a + DL_a)z + (DR_a + K_t K_b) \tag{55}$$

$$H(z) = Jz + D \tag{56}$$

Then, (49) and (50) can be rewritten as:

$$\breve{I}_a(s) = \breve{V}_a(s) \frac{\breve{H}(s)}{\breve{P}(s)} \tag{57}$$

$$\breve{\theta}(s) = \breve{V}_a(s) \frac{K_t}{s\breve{P}(s)} \tag{58}$$

Calculating the inverse Laplace transformation from (57),(58) with the use of Eqs. (12)−(14) and (22), we obtain (zero initial conditions!):

$$I_a(t>0)=\int_0^t E_a(t-s)V_a(s)ds \tag{59}$$

$$\theta(t>0)=\int_0^t E_\theta(t-s)V_a(s)ds \tag{60}$$

where

$$\begin{aligned} E_a(t>0) &= \exp(iz_1t)\times i\times res\left[\frac{\hat{H}(iz)}{\hat{P}(iz)}\right]_{z=z_1}+\exp(iz_2t)\times i\times res\left[\frac{\hat{H}(iz)}{\hat{P}(iz)}\right]_{z=z_2}= \\ &= \frac{L_aD-\alpha_1}{L_a(\alpha_1-\alpha_2)}\exp\left(-\frac{\alpha_1}{JL_a}t\right)+\frac{L_aD-\alpha_2}{L_a(\alpha_2-\alpha_1)}\exp\left(-\frac{\alpha_2}{JL_a}t\right) \end{aligned} \tag{61}$$

$$\hat{P}(iz)=-JL_az^2+i(JR_a+DL_a)z+(DR_a+K_tK_b)=-JL_a\left(z^2-i\frac{(JR_a+DL_a)}{JL_a}z-\frac{(DR_a+K_tK_b)}{JL_a}\right)=$$

$$=-JL_a(z-z_1)(z-z_2)=-JL_a\left(z-i\frac{\alpha_1}{JL_a}\right)\left(z-i\frac{\alpha_2}{JL_a}\right)$$

$$\begin{cases} z_{1,2}=i\dfrac{(JR_a+DL_a)}{2JL_a}\left(1\pm\sqrt{1-\dfrac{4JL_a(DR_a+K_tK_b)}{(JR_a+DL_a)^2}}\right)=i\dfrac{\alpha_{1,2}}{JL_a} \\ \\ \alpha_{1,2}=\dfrac{(JR_a+DL_a)}{2}\left(1\pm\sqrt{1-\dfrac{4JL_a(DR_a+K_tK_b)}{(JR_a+DL_a)^2}}\right) \end{cases}$$

$$\begin{aligned} E_\theta(t>0) &= i\times res\left[\frac{K_t}{(iz)\hat{P}(iz)}\right]_{z=0}+\exp(iz_1t)\times i\times res\left[\frac{K_t}{(iz)\hat{P}(iz)}\right]_{z=z_1}+\exp(iz_1t)\times i\times res\left[\frac{K_t}{(iz)\hat{P}(iz)}\right]_{z=z2}= \\ &= \frac{K_t}{DR_a+K_tK_b}+\frac{K_tJL_a}{(\alpha_1-\alpha_2)\alpha_1}\exp\left(-\frac{\alpha_1}{JL_a}t\right)+\frac{K_tJL_a}{(\alpha_2-\alpha_1)\alpha_2}\exp\left(-\frac{\alpha_2}{JL_a}t\right) \end{aligned} \tag{62}$$

Putting Eqs. (61),(62) into Eqs. (59),(60), we finally obtain:

$$I_a(t>0)=\frac{L_aD-\alpha_1}{L_a(\alpha_1-\alpha_2)}\int_0^t\exp\left(-\frac{\alpha_1}{JL_a}(t-s)\right)V_a(s)ds+\frac{L_aD-\alpha_2}{L_a(\alpha_2-\alpha_1)}\int_0^t\exp\left(-\frac{\alpha_2}{JL_a}(t-s)\right)V_a(s)ds \tag{63}$$

$$\theta(t>0)=\frac{K_t}{DR_a+K_tK_b}\int_0^t V_a(s)ds+\frac{K_tJL_a}{(\alpha_1-\alpha_2)\alpha_1}\int_0^t \exp\left(-\frac{\alpha_1}{JL_a}(t-s)\right)V_a(s)ds+ \\ +\frac{K_tJL_a}{(\alpha_2-\alpha_1)\alpha_2}\int_0^t \exp\left(-\frac{\alpha_2}{JL_a}(t-s)\right)V_a(s)ds \tag{64}$$

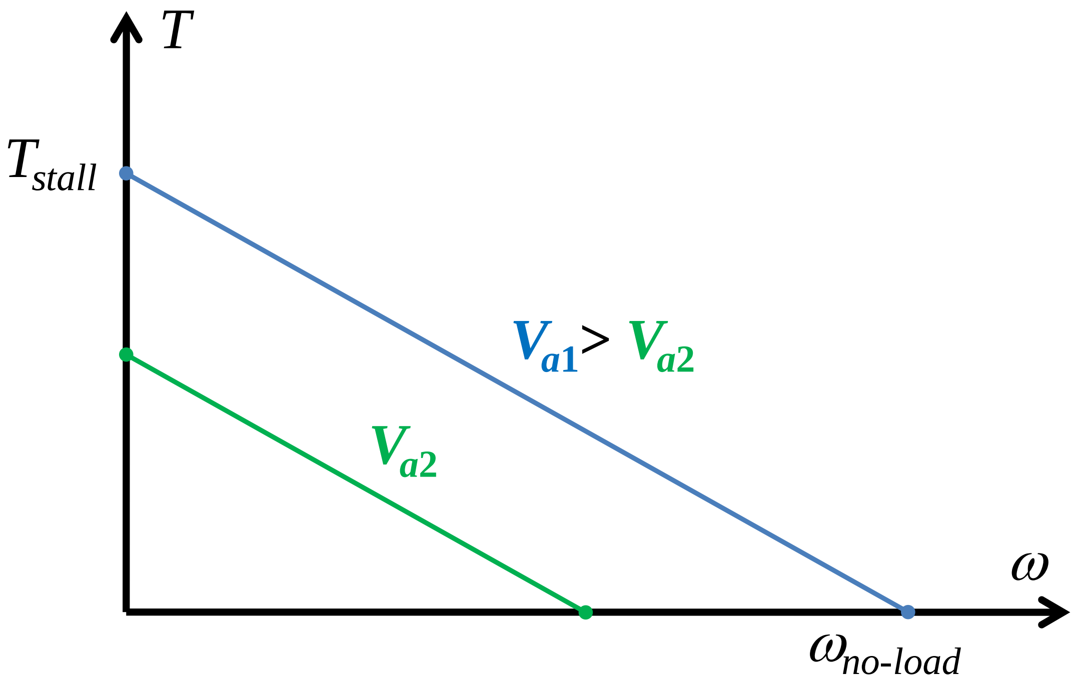
If in Eq. (45) neglect L_a (a quite reasonable assumption for a practical DC motor), then we obtain:

$$\begin{cases} R_aI_a(t)+K_b\dfrac{d\theta(t)}{dt}=V_a(t) \\ T(t)=K_tI_a(t) \end{cases} \tag{65}$$

Excluding the current $I_a(t)=\dfrac{T(t)}{K_t}$, we obtain:

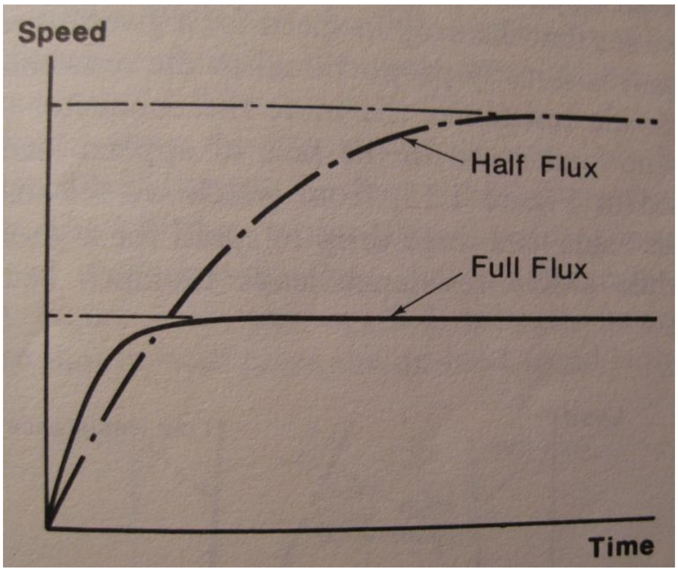
$$T(t)=\frac{V_a(t)K_t}{R_a}-\frac{K_tK_b}{R_a}\omega(t) \tag{66}$$

Eq. (66) is a straight line, $T(t)$ vs. $\omega(t)$, as shown in the figure below. If V_a is a constant voltage, after the certain characteristic time, we obtain a steady-state rotation. For this steady-state rotation, we can use the same Eq. (66).



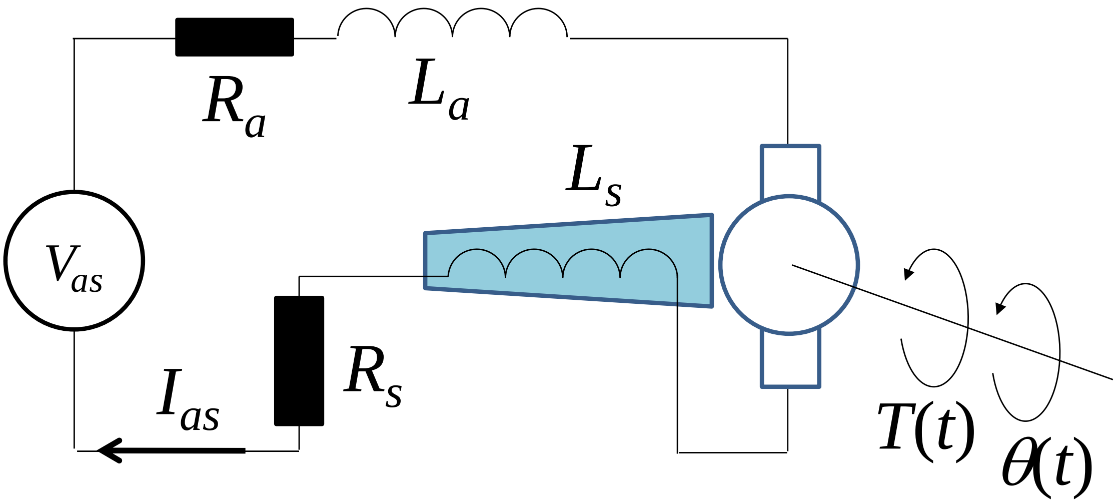
$$T_{stall}=\frac{V_aK_t}{R_a} \tag{67}$$

$$\omega_{no-load}=\frac{V_a}{K_b} \tag{68}$$

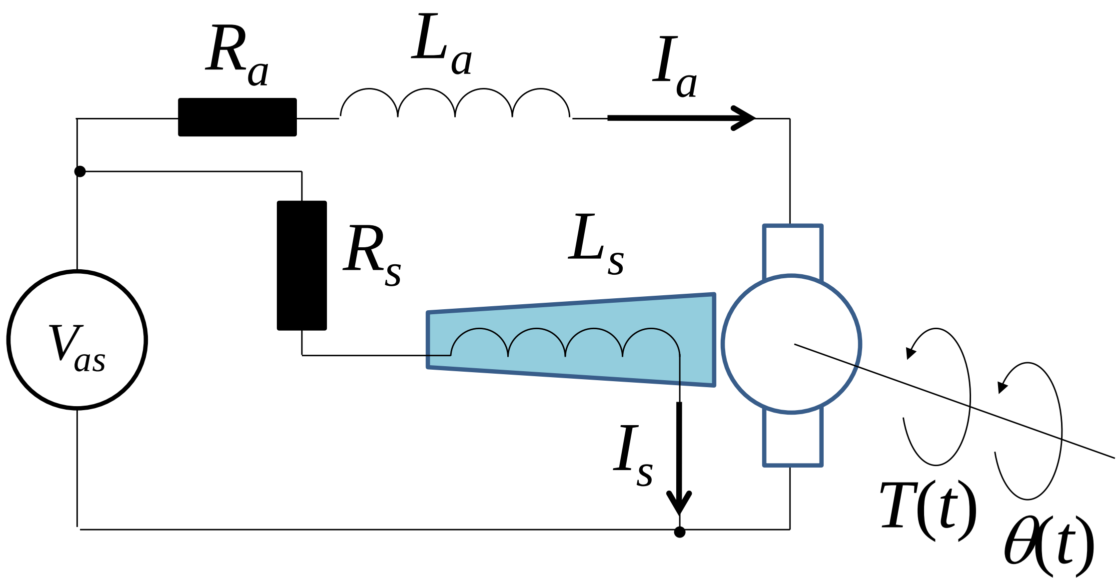


From Eq. (67),(68) we can make some important conclusions. First of all, the angular velocity is directly proportional to V_a, and this principle is used to regulate ω. Since K_b is directly proportional to the stator field, a larger stator field will reduce the rotor steady-state velocity. This conclusion is quite unexpected! However, since K_t is also directly proportional to the stator field, a larger stator field will increase the rotor torque, and hence its acceleration under a load. All these features are shown in the Figure at the left.

Above, we have considered the feeding scheme where the stator and rotor are connected to the different voltage sources. There are possible other feeding connections, where (i) the stator is connected in series with the rotor or (ii) the stator is connected in parallel with the rotor, as shown in the Figures below. These feeding connections will result in (i) a non-linear network and (ii) a non-stationary linear network, respectively. Unfortunately, these networks cannot be analysed by the analytical methods developed in our previous Lectures.



(non-linear network)



(non-stationary linear network)

V. Closed-loop speed control of a DC motor

A simple closed-loop speed control of a DC motor is shown in the Figure below. The tachometer delivers a signal $V_T(t)$ to one input of the differential amplifier and the DC reference voltage V_{ref} is applied to the other input. The differential signal $(V_{ref} - V_T(t))$ is used to maintain the rotation at constant speed. The output voltage of the differential amplifier is passed through the power amplifier to allow a bigger driving current for the motor armature. For further analysis, we will assume that the power amplifier is a current source, i.e. it provides the armature current $I_a(t)$.

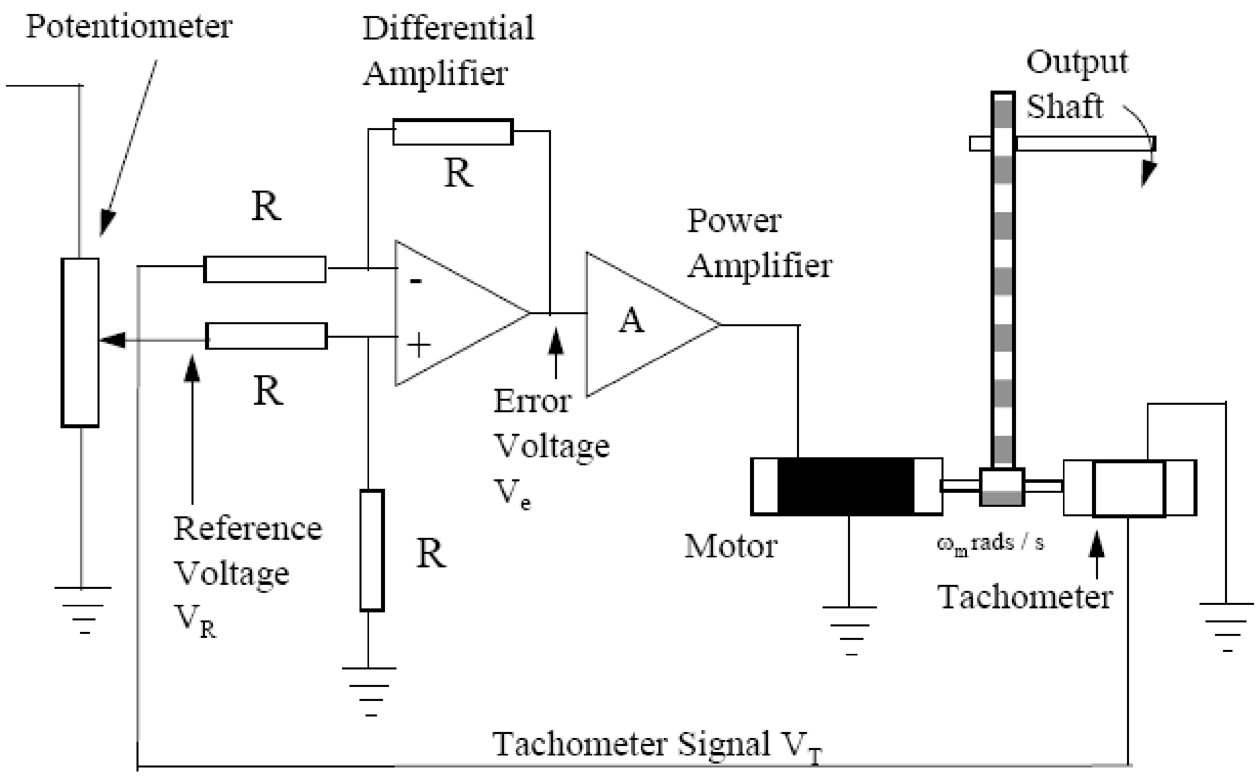
The operation of the motor can be determined from the combination of the individual transfer functions written in the s-representation (an analog of the frequency domain for the Laplace transformation). The transfer functions of the differential amplifier G_d, the power amplifier G_a, and the tachometer G_T are assumed to be constant:

$$\breve{V}_d(s) = G_d\left(\frac{V_{ref}}{s} - \breve{V}_T(s)\right) \tag{69}$$

$$\breve{I}_a(s) = G_a \breve{V}_d(s) = G_a G_d\left(\frac{V_{ref}}{s} - \breve{V}_T(s)\right) \tag{70}$$

$$\breve{V}_T(s) = G_T \breve{\omega}(s) \tag{71}$$

where V_{ref} is the DC reference voltage that defines the speed of rotation, $\left(\frac{V_{ref}}{s}\right) = V_{ref}\int\limits_0^{+\infty}\exp(-st)dt$ is the Laplace image of the DC reference voltage, and $\breve{V}_T(s)$ is the Laplace image of the tachometer voltage output, $\breve{V}_d(s)$ is the Laplace image of the differential voltage output, $\breve{I}_a(s)$ is the Laplace image of the armature current, and $\breve{\omega}(s)$ is the Laplace image of the angular velocity.



From Eqs. (53) and (54), we obtain:

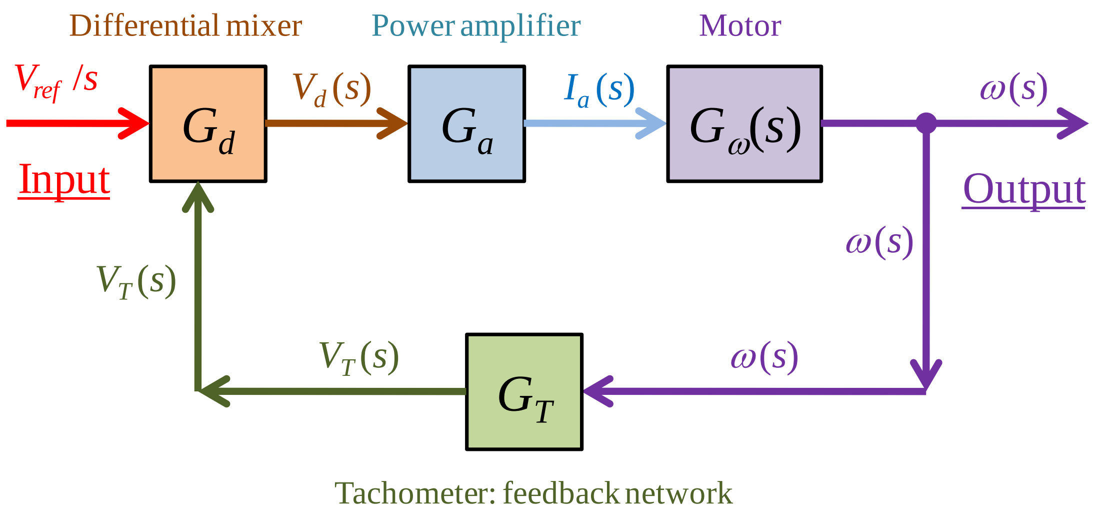
$$\breve{V}_a(s) = \breve{I}_a(s)\frac{JL_a s^2 + (JR_a + DL_a)s + (DR_a + K_t K_b)}{(Js + D)} \tag{72}$$

$$\breve{\theta}(s) = \breve{V}_a(s)\frac{K_t}{s(JL_a s^2 + (JR_a + DL_a)s + (DR_a + K_t K_b))} = \breve{I}_a(s)\frac{K_t}{s(Js + D)} \tag{73}$$

Since $\breve{\omega}(s) = \int_0^{+\infty} \frac{d\theta(t)}{dt}\exp(-st)dt = s\breve{\theta}(s)$ (where we have taken into account $\theta(0) = 0$), using Eq. (73) we can write the motor transfer function $\breve{G}_\omega(s)$:

$$\boxed{\breve{\omega}(s) = \breve{G}_\omega(s)\breve{I}_a(s) = \frac{K_t}{(Js + D)}\breve{I}_a(s)} \tag{74}$$

Let us remind that the armature current is provided by the power amplifier which is a current source. The closed-loop signal flow for our network is shown in the Figure below:



Using this signal flow graph, we obtain:

$$\breve{\omega}(s) = \frac{\breve{G}_\omega(s) G_a G_d}{1 + \breve{G}_\omega(s) G_T G_a G_d} \left(\frac{V_{ref}}{s} \right) \tag{75}$$

Therefore, we have derived the transfer function (Reference signal $\Rightarrow$ angular velocity) of the closed-loop network:

$$\breve{F}(s) = \frac{\breve{G}_\omega(s) G_a G_d}{1 + \breve{G}_\omega(s) G_T G_a G_d} \tag{76}$$

Putting Eq. (74) into Eq. (76), we finally obtain:

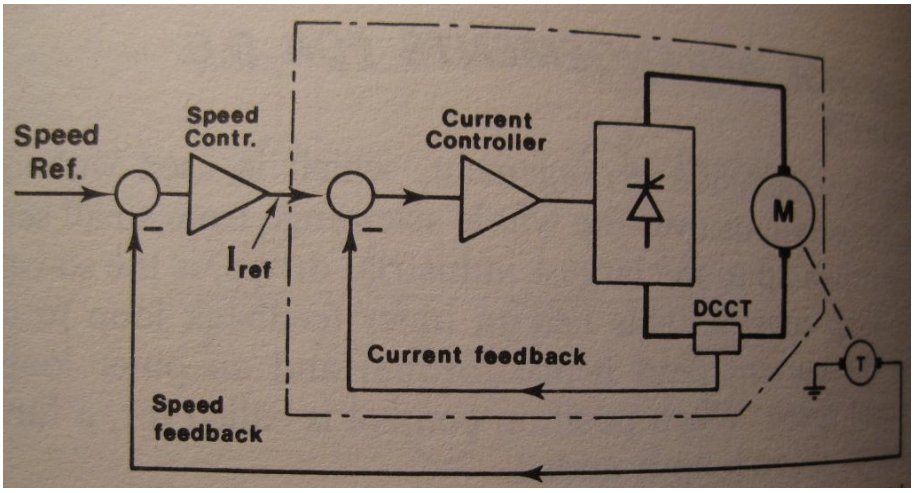
$$\breve{F}(s) = \frac{K_t G_a G_d}{Js + (D + K_t G_T G_a G_d)} \tag{77}$$

For the system stability, the only zero $z_0 = i\dfrac{(D + K_t G_T G_a G_d)}{J}$ of $\tilde{P}(z) = iJz + (D + K_t G_T G_a G_d)$ must be located in the upper complex half plain, i.e. $\dfrac{(D + K_t G_T G_a G_d)}{J} > 0$. This zero gives us the system characteristic time τ (response time):

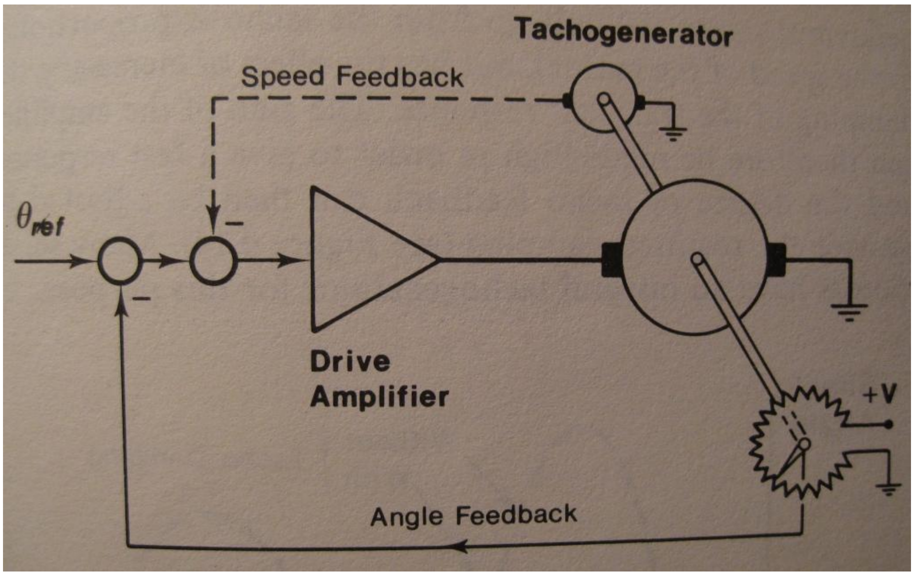
$$\tau = \frac{1}{\mathrm{Im}(z_0)} = \frac{J}{D + K_t G_T G_a G_d} \tag{78}$$

This time is directly proportional to the moment of inertia J, what is a physically obvious fact.

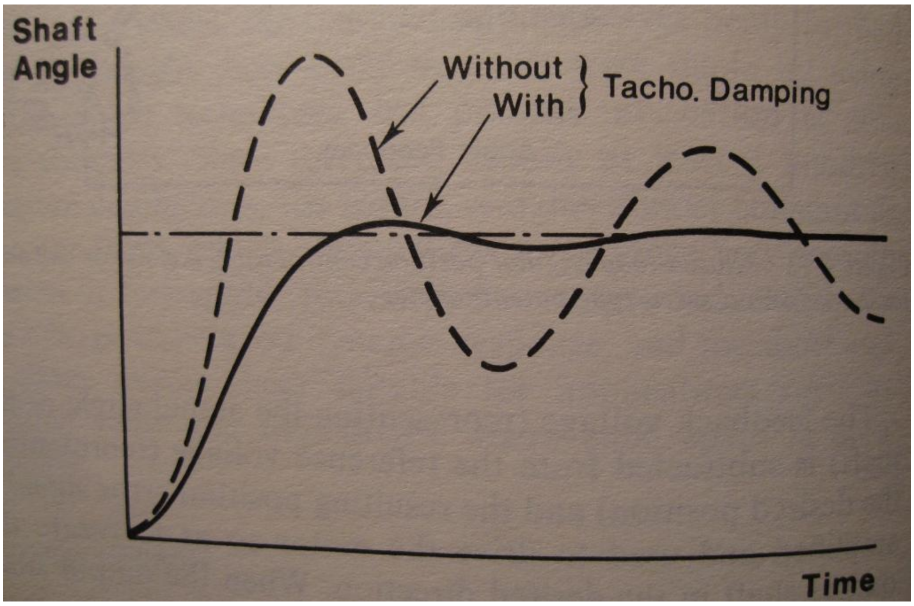
A standard DC drive system with speed and current control is shown in the Figure below. The purpose of the current loop is to make the actual motor current follow the current reference signal. It does this be comparing a feedback signal of actual motor current with the current reference signal. As long as the current control loop functions properly, the motor current can never exceed the reference value. Hence by limiting the magnitude of the current reference signal, the motor current can never exceed the specific value. It means that if for example the motor suddenly stalls because the load seizes, the armature voltage will automatically reduce to a very low value, thereby limiting the current to its maximum allowable level.



Servomotors are used in closed-loop position control applications. The potentiometer mounted on the output shaft provides a feedback voltage proportional to the actual position of the output shaft. The voltage from this potentiometer must be a linear function of angle. The feedback voltage (representing the actual angle of the shaft) is subtracted from the reference voltage (representing the desired position) and used to drive the motor so as to rotate the output shaft in the desired direction. When the output shaft reaches the target position, the position error becomes zero, no voltage is applied to the motor, and the output shaft remains at rest. Any attempt to physically move the output shaft from its target position immediately creates a position error and a restoring torque is applied by the motor. Unfortunately, the dynamic performance of the simple scheme described above is very unsatisfactory as it stands. In order to achieve a fast response and to minimize position errors caused by static friction, the gain of the amplifier needs to be high, but this in turn leads to a highly oscillatory response which is usually unacceptable. The best solution of this drawback is to use tachometer feedback in addition to the main position feedback loop, as shown in the Figure below.



Tachometer feedback has no effect on the static behaviour, but has the effect of increasing the damping of the transient response. Many servo motors have an integral tachometer for this purpose.


Shaft Angle
Without
With
Tacho. Damping
Time